# The Symbolic Method for 2-SAT

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# Exhibition result

**Theorem.** Let $S_{n,m}$ be the number of satisfiable 2-CNF with $n$ Boolean literals and $m$ clauses. Then,

$$\ddot{S}(z,w) = \left[ \sqrt{G(z,w) \odot_z \frac{1}{G(2z,w)}} \odot_z \ddot{Set}(z,w) \right] G\left(\frac{2z}{1+w}, w\right)$$

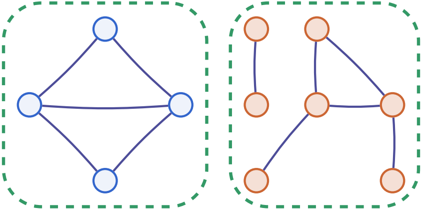
where

- $\ddot{S}(z,w) := \sum_{n=0}^{\infty} \sum_{m=0}^{2n(n-1)} S_{n,m} \frac{w^m}{(1+w)^{n^2}} \frac{z^n}{n!}$
- $\ddot{Set}(z,w) := \sum_{n=0}^{\infty} \frac{1}{(1+w)^{n^2}} \frac{z^n}{n!}$
- $G(z,w) := \sum_{n=0}^{\infty} (1+w)^{\binom{n}{2}} \frac{z^n}{n!}$ is the EGF of all simple graphs
- $\odot_z$ is the exponential Hadamard product
  $\left(\sum_{n=0}^{\infty} a_n(w) \frac{z^n}{n!}\right) \odot_z \left(\sum_{n=0}^{\infty} b_n(w) \frac{z^n}{n!}\right) := \sum_{n=0}^{\infty} a_n(w) b_n(w) \frac{z^n}{n!}$.

Part I. Back to the origin: generating functions

# The cartesian product

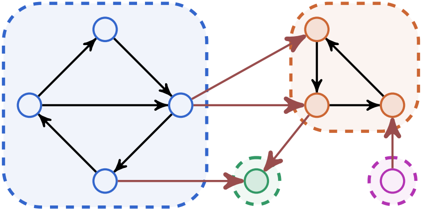
$$\left(a_0 + a_1\frac{z}{1!} + a_2\frac{z^2}{2!} + \dots\right)\left(b_0 + b_1\frac{z}{1!} + b_2\frac{z^2}{2!} + \dots\right) = c_0 + c_1\frac{z}{1!} + c_2\frac{z^2}{2!} + \dots$$

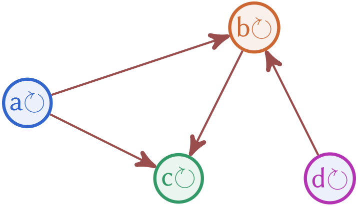


The convolution rule corresponding to EGF:

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

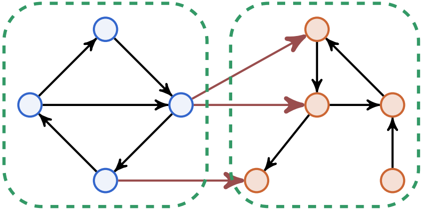
# Directed graphs and their components






- Components a⟳ , b⟳ , c⟳ and d⟳ are *strongly-connected* components.
- Components a⟳ and d⟳ are *source-like* components
- Component c⟳ is a *sink-like* component

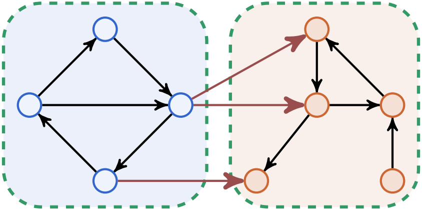
# The arrow product



# The arrow product convolution rule

Let $a_n = a_n(w)$, $b_n = b_n(w)$, $c_n = c_n(w)$,

$$\left(\sum_{n=0}^{\infty} a_n \frac{1}{(1+w)^{\binom{n}{2}}}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{1}{(1+w)^{\binom{n}{2}}}\frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{1}{(1+w)^{\binom{n}{2}}}\frac{z^n}{n!}$$



The convolution rule corresponding to Graphic GF:

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} (1+w)^{k(n-k)}$$

# Conversion between Exponential GF and Graphic GF

$$\widehat{A}(z,w) = \sum_{n=0}^{\infty} a_n(w) \frac{1}{(1+w)^{\binom{n}{2}}} \frac{z^n}{n!}, \quad A(z,w) = \sum_{n=0}^{\infty} a_n(w) \frac{z^n}{n!}$$

- **Exponential Hadamard product:**

$$\left( \sum_{n \geqslant 0} a_n(\mathrm{w}) \frac{z^n}{n!} \right) \odot_z \left( \sum_{n \geqslant 0} b_n(\mathrm{w}) \frac{z^n}{n!} \right) := \sum_{n \geqslant 0} a_n(\mathrm{w}) b_n(\mathrm{w}) \frac{z^n}{n!}$$

- **Exponential GF for graphs, Graphic GF for sets:**

$$G(z,w) = \sum_{n \geqslant 0} (1+w)^{\binom{n}{2}} \frac{z^n}{n!}, \quad \widehat{Set}(z,w) = \sum_{n \geqslant 0} \frac{1}{(1+w)^{\binom{n}{2}}} \frac{z^n}{n!},$$

- **Conversion formulas:**

$$\boxed{A(z,w) = G(z,w) \odot_z \widehat{A}(z,w)} \quad \boxed{\widehat{A}(z,w) = \widehat{Set}(z,w) \odot_z A(z,w)}$$

# P.S. Analytic conversion principle (Graphic ↔ Exponential)

e.g. [Flajolet, Salvy, Schaffer '2004] and other works on the exponents of quadratic forms

$$\widehat{A}(z,w) = \sum_{n=0}^{\infty} a_n(\mathrm{w}) \frac{1}{(1+w)^{\binom{n}{2}}} \frac{z^n}{n!}, \quad A(z,w) = \sum_{n=0}^{\infty} a_n(\mathrm{w}) \frac{z^n}{n!}$$

- **Fourier integral:**

$$e^{-t^2/2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt} e^{-x^2/2} dx.$$

- **Exponential to Graphic:**

$$\widehat{A}(z,w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} A\Big(z\sqrt{1+w}\, e^{ix\sqrt{\log(1+w)}}, w\Big) e^{-x^2/2} dx.$$

*Knowing that such a conversion exists is an important step for refined asymptotic analysis in the future. We have already used it in [de Panafieu, D., Ralaivaosaona, Rasendrahasina, Wagner '2021+].*
https://arxiv.org/abs/2009.12127

# Part II. Families of directed graphs

# Main enumeration theorem

[Liskovets, Robinson, Gessel, Wright et. al. '1970's] [de Panafieu, D. '2019]

## Theorem (rediscovery of the results from '1970s)

- *Graphic GF for digraphs with strongly connected components from given family $\mathcal{SCC}$ is*

$$\widehat{D}(z, w) = \frac{1}{e^{-\mathrm{SCC}(z,w)} \odot_z \widehat{Set}(z, w)}$$

*where* $\mathrm{SCC}(z, w)$ *is the Exponential GF.*

## Compare with simple graphs (folklore)

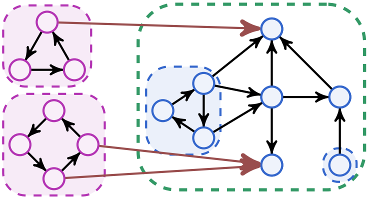
- Exponential GF for graphs with connected components from given family $\mathcal{C}$ is

$$G(z, w) = e^{C(z,w)} = \frac{1}{e^{-C(z,w)}}$$

# Proof of the main enumeration theorem

- Let u mark *source-like components* in $\mathcal{D}$.
- $\mathcal{D}$ with *distinguished* source-like components is an **arrow product** of a *set* of strong components and $\mathcal{D}$.

$$\widehat{D}(z, w, \mathrm{u} + \mathbf{1}) = \left(e^{\mathrm{u}\cdot \mathbf{SCC}(z,w)} \odot_z \widehat{Set}(z, w)\right) \cdot \widehat{D}(z, w, 1).$$



are distinguished source-like

are other source-like components.

- Set u $= -1$. Result follows: $\widehat{D}(z, w) = \dfrac{1}{e^{-\mathbf{SCC}(z,w)} \odot_z \widehat{Set}(z, w)}$.

# Corollary: strongly connected digraphs

[Liskovets, Robinson, Gessel, Wright et. al. '1970's] [de Panafieu, D. '2019]

## Theorem

*Exponential GF of strongly connected digraphs is*

$$\mathrm{SCC}(z, w) = -\log\left(G(z, w) \odot_z \frac{1}{G(z, w)}\right)$$

**Proof.** Inversion of the main enumeration theorem

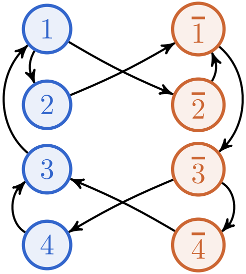
$$G(z, w) = \widehat{D}(z, w) = \frac{1}{e^{-\mathrm{SCC}(z,w)} \odot_z \widehat{Set}(z, w)}$$

Graphic GF of all digraphs $\widehat{D}(z, w)$ equals the EGF of graphs $G(z, w)$.

Part III. Symbolic method for 2-CNF via implication digraphs

# Implication digraphs

$$\begin{cases} \neg x_1 \lor \neg x_2 & = 1 \\ \neg x_1 \lor x_2 & = 1 \\ x_3 \lor x_4 & = 1 \\ x_1 \lor \neg x_3 & = 1 \\ x_3 \lor \neg x_4 & = 1 \end{cases}$$




Replace each clause $x \lor y$ with two implications $\bar{x} \to y$ and $\bar{y} \to x$.

## Proposition (folklore / [Aspvall, Plass, Tarjan '82])

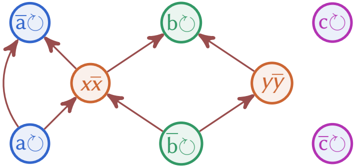
2-CNF is satisfiable if and only if there is no *contradictory circuit*.

## The above 2-CNF is not satisfiable

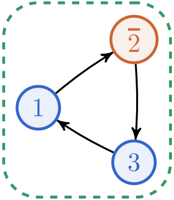
$$1 \longrightarrow \bar{2} \longrightarrow \bar{1} \longrightarrow \bar{3} \longrightarrow \bar{4} \longrightarrow 3 \longrightarrow 1$$

**N.B.** Each variable of a contradictory component belongs to a contradictory circuit.

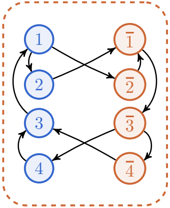
# Implication digraphs and their components




**2-CNF implication digraph**



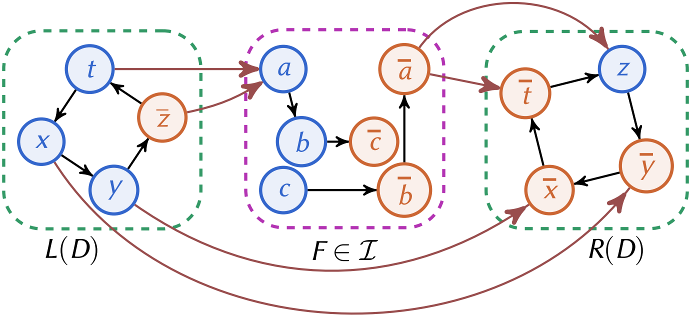

**(Ordinary)**




**(Contradictory)**

- $x\bar{x}$ and $y\bar{y}$ are *contradictory* strongly connected
- a⟳ and b̄⟳ are *ordinary* source-like
- ā⟳ and b⟳ are *ordinary* sink-like
- c⟳ and c̄⟳ are *ordinary isolated*

# The implication product

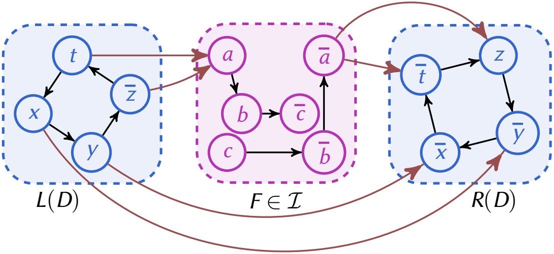


# The implication product convolution rule

If $\widehat{A}$ is Graphic GF and $\ddot{B}, \ddot{C}$ are Implication GF then

$$\Big(\sum_{n=0}^{\infty} a_n \frac{2^n}{(1+w)^{\binom{n+1}{2}}} \frac{z^n}{n!}\Big)\Big(\sum_{n=0}^{\infty} b_n \frac{1}{(1+w)^{n^2}} \frac{z^n}{n!}\Big) = \sum_{n=0}^{\infty} c_n \frac{1}{(1+w)^{n^2}} \frac{z^n}{n!}$$

$$\widehat{A}\Big(\tfrac{2z}{1+w}\Big) \cdot \ddot{B}(z,w) = \ddot{C}(z,w)$$




Combinatorial convolution rule corresponding to Implication GF:

$$c_n = \sum_{k=0}^{n} \binom{n}{k} 2^k a_k b_{n-k} (1+w)^{k \cdot 2(n-k) + \binom{k}{2}}$$

# Main enumeration theorem for 2-CNF

Theorem ([de Panafieu, D., Ravelomanana ’2021])

- *Implication GF for implication digraphs with ordinary components from given $\mathcal{SCC}$ and contradictory components from given $\mathcal{CSCC}$ is*

$$\ddot{CNF}_2(z,w) = \frac{e^{\mathrm{CSCC}(z,w) - \mathrm{SCC}(2z,w)/2} \odot_z \ddot{Set}(z,w)}{e^{-\mathrm{SCC}\left(\frac{2z}{1+w},w\right)} \odot_z \widehat{Set}(z,w)}$$

- *where*

$$\widehat{Set}(z,w) = \sum_{n \geqslant 0} \frac{1}{(1+w)^{\binom{n}{2}}} \frac{z^n}{n!}, \quad \ddot{Set}(z,w) = \sum_{n \geqslant 0} \frac{1}{(1+w)^{n^2}} \frac{z^n}{n!}.$$

# Proof of the main enumeration theorem

- Let u mark *ordinary source-like components* in 2-CNF.
- Let v mark *ordinary isolated components* in 2-CNF (by pairs).
- Take **implication product** of *set* of ordinary components and 2-CNF.
- Add an arbitrary subset of ordinary isolated components. EGF of one pair of isolated components is $\mathrm{SCC}(2z)/2$.
- Now, every source-like component is marked by u or 1, and every ordinary isolated pair is marked by 2u, v or 1.

$$egf[\ddot{\mathrm{CNF}}_2(z, \mathsf{u}+1, 2\mathsf{u}+\mathsf{v}+1)] \\ = egf\Big[\Big(e^{\mathsf{u}\cdot\mathrm{SCC}\left(\frac{2z}{w+1}\right)} \odot_z \widehat{Set}(z)\Big)\ddot{\mathrm{CNF}}_2(z,1,1)\Big] \cdot e^{\mathsf{v}\mathrm{SCC}(2z)/2}$$

- Let $\mathsf{u} = -1$. **An implication digraph without source-like ordinary components is a disjoint set of contradictory and ordinary components.**
- Complete with arithmetic transformations.

# Two corollaries

## Corollary 1 (inversion of the main theorem)

Exponential GF of contradictory strongly connected components is

$$\mathrm{CSCC}(z, w) = \frac{1}{2}\mathrm{SCC}(2z, w) + \log\left(\mathrm{BG}(z, w) \odot_z \left[\frac{\ddot{\mathrm{CNF}}_2(z, w)}{G\left(\frac{2z}{w+1}\right)}\right]\right)$$

where $\mathrm{BG}(z) = \sum_{n=0}^{\infty}(1 + w)^{n^2}\dfrac{z^n}{n!}$ is the EGF of bipartite graphs.

## Corollary 2 (no contradictory components)

Implication GF of satisfiable 2-CNF is

$$\ddot{S}(z, w) = \left[\sqrt{G(z, w) \odot_z \frac{1}{G(2z, w)}} \odot_z \ddot{Set}(z, w)\right] G\left(\frac{2z}{1 + w}, w\right).$$

# Summary

## Discussed today

1. New symbolic method for 2-CNF with given strong and contradictory components: Implication GF and friends
2. Satisfiable 2-CNF: particular case with $\mathrm{CSCC}(z, w) = 0$.
3. Exponential GF for Contradictory SCC

## Behind the scenes

- 2-CNF with marked source-like SCC, isolated SCC and contradictory SCC
- Multi-2-CNF (with loops and multiple clauses) yield simpler expressions
- Asymptotic analysis of divergent series (Fourier and friends)

This talk is dedicated to the memory of Philippe Flajolet

and to the strongly connected people of contradictory Belarus

Thank you for your attention.