# Symbolic method for directed graphs

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# Outline of the current talk

$$\underbrace{\underbrace{\text{Symbolic method}}_{\text{Part I}}\ \text{for directed graphs}}_{\text{Part II}}$$

# Part I. Symbolic method

# Generating functions and the symbolic method

## Definition

The **ordinary generating function** of the sequence

$$a_0, a_1, a_2, \ldots$$

is a formal power series

$$A(z) := a_0 + a_1 z + a_2 z^2 + \ldots$$

## Example

Let $a_n$ = number of connected *unlabelled* graphs with $n$ vertices:

$$(a_n)_{n=0}^{\infty} := (1, 1, 1, 2, 6, 21, 112, 853, 11117, 261080, 11716571, \ldots)$$

N.B.: Formal power series $A(z)$ is divergent for any $|z| > 0$.

# Generating functions and the symbolic method

## Definition

The **exponential generating function** of the sequence

$$a_0, a_1, a_2, \ldots$$

is a formal power series

$$A(z) := a_0 + a_1 \frac{z}{1!} + a_2 \frac{z^2}{2!} + \ldots$$

## Example

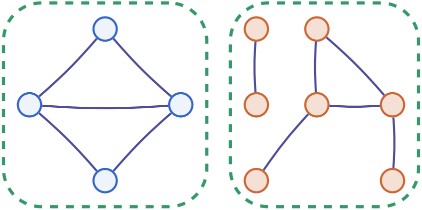
Let $a_n =$ number of connected *labelled* graphs with $n$ vertices:

$$(a_n)_{n=0}^{\infty} := (1, 1, 1, 4, 38, 728, 26704, 1866256, 251548592, \ldots)$$

N.B.: Formal power series $A(z)$ is divergent for any $|z| > 0$.

# Labelled cartesian product

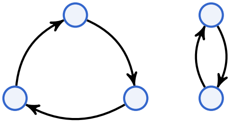
$$A(z) = a_0 + a_1 \frac{z}{1!} + a_2 \frac{z^2}{2!} + \ldots, \quad B(z) = b_0 + b_1 \frac{z}{1!} + b_2 \frac{z^2}{2!} + \ldots$$



| **coefficient level** | **exponential GF level** |
|---|---|
| $c_n := \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ | $C(z) = A(z) \cdot B(z)$ |

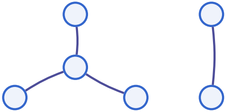
# Two examples

## Example (A permutation is a set of cycles)



- $\dfrac{1}{1-z} = 1 + C(z) + \dfrac{C^2(z)}{2!} + \ldots = e^{C(z)}$
- $C(z) = z + 1!\dfrac{z^2}{2!} + 2!\dfrac{z^2}{3!} + \ldots = \log\dfrac{1}{1-z}$

## Example (Connected labelled graphs)



- All graphs $G(z) = \sum_{n\geq 0} 2^{\binom{n}{2}} \dfrac{z^n}{n!}$
- Graphs are sets of connected graphs

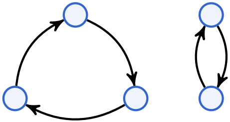
$$G(z) = \exp(C(z))$$

- $C(z) = \log\left(\sum_{n\geq 0} 2^{\binom{n}{2}} \dfrac{z^n}{n!}\right)$

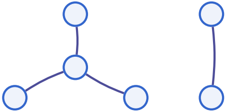
# Two examples

## Example (A permutation is a set of cycles)

- $F(z, u) = 1 + uC(z) + u^2 \frac{C^2(z)}{2!} + \ldots = e^{uC(z)}$
- $C(z) = z + 1! \frac{z^2}{2!} + 2! \frac{z^2}{3!} + \ldots = \log \frac{1}{1-z}$



## Example (Connected labelled graphs)



- All graphs $G(z, u) = \sum_{n \geq 0} (1 + u)^{\binom{n}{2}} \frac{z^n}{n!}$
- Graphs are sets of connected graphs

$$G(z, u) = \exp(C(z, u))$$

- $C(z, u) = \log \left( \sum_{n \geq 0} (1 + u)^{\binom{n}{2}} \frac{z^n}{n!} \right)$

# Combinatorial operations

| Operation | interpretation |
|---|---|
| $A(z) + B(z)$ | disjoint union |
| $A(z)B(z)$ | cartesian product |
| $\exp(A(z))$ | set |
| $A(B(z))$ | substitution |
| $z\partial_z A(z)$ | pointing |
| $A(z) \odot B(z)$ | Hadamard product |
| $\dots$ | $\dots$ |

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**The philosophy of the symbolic method (Bergeron, Labelle, Leroux).**

$$\boxed{\text{Decomposition}} \quad \Rightarrow \quad \boxed{\text{Equation}}$$

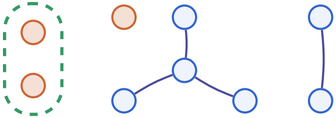
**Follow-up: asymptotic analysis (Flajolet, Odlyzko, …).**

$$\boxed{\text{Equation}} \quad \Rightarrow \quad \boxed{\text{Asymptotics}}$$

# Inclusion-exclusion principle

Additional variables mark special vertices or groups of vertices

$$A(z, \mathrm{w}, \mathrm{u}) = \sum_{n,k,r} a_{n,k,r} \mathrm{w}^k \mathrm{u}^r \frac{z^n}{n!}$$



**Example:**

$a_{n,k,r} = \#_{\text{of}}$ graphs with

- $n$ vertices
- $k$ edges
- $r$ isolated vertices

$$B(z, \mathrm{w}, \mathrm{u}) := A(z, \mathrm{w}, \mathrm{u} + 1) = \sum_{n,k,r} b_{n,k,r} \mathrm{w}^k \mathrm{u}^r \frac{z^n}{n!}$$

$b_{n,k,r} = \#_{\text{of}}$ graphs with

- $n$ vertices
- $k$ edges
- $r$ *distinguished* isolated vertices

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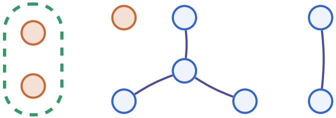

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$b_{n,k,r} = \#_{\text{of}}$ graphs with

- $n$ vertices
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- $r$ *distinguished* isolated vertices

Questions?

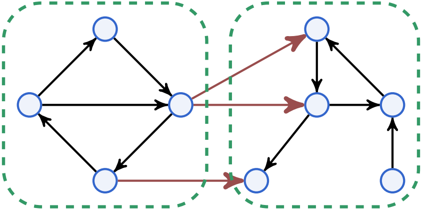
# Part II. Symbolic method for directed graphs

# Known convolution rules

| **convolution** | **generating function** | **name** |
| --- | --- | --- |
| $\sum_k a_k b_{n-k}$ | $A(z) := \sum_{n\geq 0} a_n z^n$ | ordinary GF |
| $\sum_k \binom{n}{k} a_k b_{n-k}$ | $A(z) := \sum_{n\geq 0} a_n \frac{z^n}{n!}$ | exponential GF |
| $\sum_d a_d b_{n/d}$ | $A(s) := \sum_{n\geq 1} \frac{a_n}{n^s}$ | Dirichlet series |
| $\sum_k \binom{n}{k} a_k b_{n-k} 2^{k(n-k)}$ | $A(z) := \sum_{n\geq 0} a_n \frac{z^n}{2^{\binom{n}{2}} n!}$ | graphic GF |

# Graphic convolution rule

$$A(z) = a_0 + \frac{a_1 z}{1!2^{\binom{1}{2}}} + \frac{a_2 z^2}{2!2^{\binom{2}{2}}} + \dots, \quad B(z) = b_0 + \frac{b_1 z}{1!2^{\binom{1}{2}}} + \frac{b_2 z^2}{2!2^{\binom{2}{2}}} + \dots$$



| **coefficient level** | **graphic GF level** |
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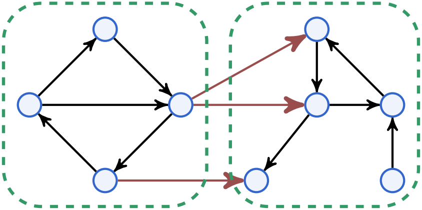
| $c_n := \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} 2^{k(n-k)}$ | $C(z) = A(z) \cdot B(z)$ |

# Graphic convolution rule

$$A(z, w) = a_0 + \frac{a_1(w)z}{1!(1+w)^{\binom{1}{2}}} + \frac{a_2(w)z^2}{2!(1+w)^{\binom{2}{2}}} + \frac{a_3(w)z^3}{3!(1+w)^{\binom{3}{2}}} + \dots$$



| **coefficient level** | **graphic GF level** |
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| $c_n := \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} 2^{k(n-k)}$ | $C(z) = A(z) \cdot B(z)$ |
| $c_n := \sum_{k=0}^{n} \binom{n}{k} a_k(w) b_{n-k}(w) (1+w)^{k(n-k)}$ | $C(z, w) = A(z, w) B(z, w)$ |

$w$ marks edges

# Conversion between exponential GF and graphic GF

$$A(z) = a_0 + a_1 \frac{z}{1!} + a_2 \frac{z^2}{2!} + \dots,$$

$$\widehat{A}(z) = a_0 + a_1 \frac{z}{1!(1+w)^{\binom{1}{2}}} + a_2 \frac{z^2}{2!(1+w)^{\binom{2}{2}}} + \dots$$

- **Exponential Hadamard product:**

$$\left(\sum_{n\geq 0} a_n \frac{z^n}{n!}\right) \odot \left(\sum_{n\geq 0} b_n \frac{z^n}{n!}\right) := \sum_{n\geq 0} a_n b_n \frac{z^n}{n!}$$

- **Exponential GF for graphs, graphic GF for sets:**

$$G(z, w) = \sum_{n\geq 0} (1+w)^{\binom{n}{2}} \frac{z^n}{n!}, \quad \widehat{Set}(z, w) = \sum_{n\geq 0} \frac{1}{(1+w)^{\binom{n}{2}}} \frac{z^n}{n!},$$

- **Conversion formulas:**

$$\boxed{A(z, w) = G(z, w) \odot \widehat{A}(z, w)} \quad \boxed{\widehat{A}(z, w) = \widehat{Set}(z, w) \odot A(z, w)}$$

# Main results

## Theorem (1, rediscovery of Liskovets' result)

- *Exponential GF for strongly connected digraphs*

$$SCC(z, w) = -\log\left(G(z, w) \odot \frac{1}{G(z, w)}\right)$$

## Compare

- Exponential GF for connected graphs

$$C(z, w) = -\log\left(\frac{1}{G(z, w)}\right)$$

# Main results

## Theorem (2, rediscovery of Robinson's result)

- *Graphic GF for digraphs with strongly connected components from given family*

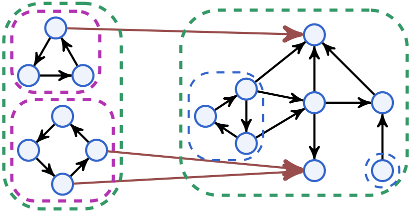
$$\widehat{D}(z,w) = \frac{1}{\widehat{Set}(z,w) \odot e^{-SCC(z,w)}}$$

## Compare

- Exponential GF for graphs with connected components from given family

$$G(z,w) = \frac{1}{e^{-C(z,w)}}$$

# Short proof of Theorem 2



$$\widehat{D}(z, w, s + 1) = \left(\widehat{Set} \odot e^{s \cdot SCC(z,w)}\right) \cdot \widehat{D}(z, w, 1)$$

s marks source-like strongly connected components

- Put $s = -1$

$$1 = \left(\widehat{Set} \odot e^{-SCC(z,w)}\right) \cdot \widehat{D}(z, w, 1)$$

- The result follows:

$$\widehat{D}(z, w, 1) = \frac{1}{\widehat{Set} \odot e^{-SCC(z,w)}}$$

# Why rediscovering existing results?

**We need more:**

- New refined combinatorial decompositions
- New analytic tools for the new symbolic method

**New expected applications:**

1. Digraphs with degree constraints
2. Asymptotics of directed acyclic graphs
3. Phase transition in directed graphs
4. Asymptotics of strongly connected graphs
5. Phase transition in 2-SAT and asymptotics of contradictory strong components

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Thank you for your attention! Questions?