Subcritical phases of random structures

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Based on joint works with
Élie De Panafieu and **Vlady Ravelomanana**

and including a very recent work of
Dimbinaina Ralaivaosaona, **Vonjy Rasendrahasina** and **Stephan Wagner**

Université Paris-13

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Outline of the current talk

Subcritical phases of random structures

Part I (Subcritical phases)

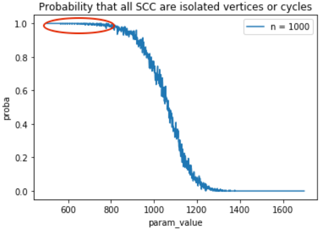
Part II (Subcritical phases of random structures)

Part I:

- Graphs

Part II:

- Graphs with degree constraints
- 2-SAT
- Acyclic digraphs
- Digraphs



Part I. Phase transition in a random graph

Evolution of random graphs

Random graph G with n vertices and m edges.

- $m = o(n^{1/2})$ isolated vertices and edges

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- $m = \frac{1}{2}n$ complex connected components appear (not trees, not unicycles). **Maximal component size** $\Theta(n^{2/3})$

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- $m = \frac{1}{2}cn$, $c > 1$ unique giant component of size $\Theta(n)$. Second largest component $O(\log n)$
- $m = \frac{1}{2}n \log n$ graph becomes connected

Naming the phases

$$m = cn, \quad c = \frac{1}{2}(1 + \mu n^{-1/3})$$

very subcritical	$c \leqslant \frac{1}{2} - \epsilon$
subcritical	$\mu \to -\infty$
critical	$\mu \in \mathbb{R}$
supercritical	$\mu \to +\infty$
very supercritical	$c \geqslant \frac{1}{2} + \epsilon$

Subcritical phase: a good starting point to understand other combinatorial structures.

Simplest phase transition ever

Theorem

Let $c > 0$. As $n \to \infty$,

$$c^n \to \begin{cases} 0, & c < 1 - \epsilon; \\ e^x, & c = 1 + \frac{x}{n}; \\ \infty, & c > 1 + \epsilon. \end{cases}$$

Critical window: $c = 1 + \frac{x}{n}$, $x \in \mathbb{R}$.

Bonus: asymptotic behaviour when $x = x(n)$, $x \to \infty$?

Phase transition in random graphs

In this talk, all the objects are labelled.

Theorem

Consider a random graph G with n vertices and m edges

$$m = cn$$

Then, as $n \to \infty$,

$$\mathbb{P}(G \text{ consists of trees and unicycles}) \to \begin{cases} 1, & c < \frac{1}{2} - \epsilon; \\ f(\mu), & c = \frac{1}{2}(1 + \mu n^{-1/3}); \\ 0, & c > \frac{1}{2} + \epsilon. \end{cases}$$

Phase transition in random graphs

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Proof.

- Let $T(z, w)$ and $U(z, w)$ be the EGF of unrooted trees and unicycles.
- Compute $n![z^n w^m]e^{T(z,w)+U(z,w)}$ using saddle point method and divide by the total number of graphs.

□

Going into more details about $f(\mu)$

$$\mathbb{P}(G \text{ consists of trees and unicycles}) \to \begin{cases} 1, & c < \frac{1}{2} - \epsilon; \\ f(\mu), & c = \frac{1}{2}(1 + \mu n^{-1/3}); \\ 0, & c > \frac{1}{2} + \epsilon. \end{cases}$$

$f(\mu)$ is completely known ([Flajolet, Janson, Knuth, Luczak, Pittel]):

$$f(\mu) = \sqrt{\frac{2\pi}{3}} e^{-\mu^3/6} \sum_{k \geqslant 0} \frac{\left(\frac{1}{2} 3^{2/3} \mu\right)^k}{k! \Gamma\left(\frac{1}{2} - \frac{2}{3} k\right)}$$

Asymptotics at the tails is also **known**:

$$f(\mu) \sim \begin{cases} 1 - \dfrac{5}{24|\mu|^3}, & \mu \to -\infty; \\ \dfrac{\sqrt{2\pi}}{\Gamma(\frac{1}{4})} \dfrac{e^{-\mu^3/6}}{2^{1/4} \mu^{3/4}}, & \mu \to +\infty. \end{cases}$$

Going further down the rabbit hole about $f(\mu)$

When $\mu \to -\infty$,

$$f(\mu) \sim 1 - \frac{5}{24|\mu|^3}, \quad \mu \to -\infty.$$

Question. What is so special about $\frac{5}{24}$?

When $\mu \to +\infty$, **only when** $|\mu| \ll n^{1/12}$,

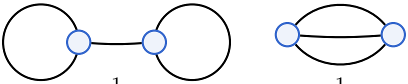
$$f(\mu) \sim \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{4})} \frac{e^{-\mu^3/6}}{2^{1/4}\mu^{3/4}}, \quad \mu \to +\infty$$

Question. What is so special about $n^{1/12}$?

Heuristical explanation of $\frac{5}{24}$

$$\mathbb{P}(\text{only trees and unicycles}) \sim 1 - \frac{5}{24|\mu|^3}, \quad \mu \to -\infty.$$

The first non-unicyclic components appearing are connected graphs with 2 cycles. Cubic multigraphs with 2 vertices with weights:



$\frac{1}{4}$ $\frac{1}{6}$

The sum of the "compensation factors"

$$\frac{1}{2!}\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{5}{24}$$

Learning more about $\frac{5}{24}$

The (weighted) number of cubic multigraphs with $2r$ vertices and $3r$ edges is

$$e_r = \frac{(6r)!}{(2r)!(3r)!2^{5r}3^{2r}}, \quad e_1 = \frac{5}{24}$$

Theorem

As $m = \frac{1}{2}n$, $n \to \infty$,

$$\mathbb{P}(\textit{complex component has } \text{excess } r) \sim \sqrt{\frac{2\pi}{3}} \cdot \frac{e_r}{3^r\Gamma(r + \frac{1}{2})}.$$

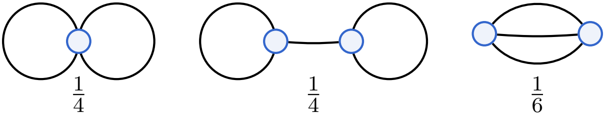
Moral. Subcritical phase is helpful for understanding of the combinatorics inside the window.

Heuristical explanation of $|\mu| \ll n^{1/12}$

As $m = \frac{n}{2}(1 + \mu n^{-1/3})$,

$$\mathbb{P}(G \text{ consists of trees and unicycles}) = f(\mu) + O(\mu^4 n^{-1/3})$$

the error term $O(\mu^4 n^{-1/3})$ corresponding to non-cubic kernels.



When $\mu = \Theta(n^{1/12})$, kernels with degrees $\geqslant 4$ appear with positive probability.

Part II. Beyond random graphs

Graph-like combinatorial structures

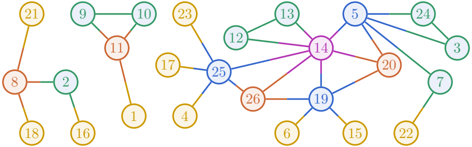
- Graphs with degree constraints
- 2-SAT
- Acyclic digraphs
- Digraphs

Graphs with allowed degrees from a given set

Graphs with degree constraints

[De Panafieu, Ramos], [D., Ravelomanana]

Allowed degrees: $\Delta = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \ldots\}, \quad \delta_1 = 1$ or $\delta_2 = 1$.



Theorem (D., Ravelomanana '2018)

As $n \to \infty$, for $m = cn$,

$$\mathbb{P}(G_\Delta \text{ consists of trees and unicycles}) \to \begin{cases} 1, & c < \alpha_\Delta - \epsilon; \\ f_\Delta(\mu), & c = \alpha_\Delta(1 + \mu n^{-1/3}); \\ 0, & c > \alpha_\Delta + \epsilon. \end{cases}$$

$f_\Delta(\mu) = f(C_\Delta \mu)$, where α_Δ and C_Δ are explicit.

Graphs with degree constraints

[De Panafieu, Ramos], [D., Ravelomanana]

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$f_\Delta(\mu) = f(C_\Delta \mu)$, where α_Δ and C_Δ are explicit.

Proof.

- Let $T_\Delta(z, w)$ and $U_\Delta(z, w)$ be the EGF of unrooted trees and unicycles with degree constraints.
- Compute $n![z^n w^m]e^{T_\Delta(z,w)+U_\Delta(z,w)}$ and divide by total number of graphs (given by [De Panafieu, Ramos])

□

Subcritical phase of 2-SAT

2-SAT: Some background

Theorem (Bollobas, Borgs, Chayes, Kim, Wilson '2001)

Consider a random 2-CNF $F_{n,m}$ with n variables and $m = cn$ clauses.

$$\mathbb{P}(F_{n,m} \text{ is SAT}) \to \begin{cases} 1 - \Theta(|\mu|^{-3}), & \mu \to -\infty; \\ \Theta(1), & c = 1 + \mu n^{-1/3}; \\ \exp(-\Theta(|\mu|^3)), & \mu \to \infty. \end{cases}$$

Theorem (Kim '2008)

$$\mathbb{P}(F_{n,m} \text{ is SAT}) \sim 1 - \frac{1}{16|\mu|^3}, \quad \mu \to -\infty.$$

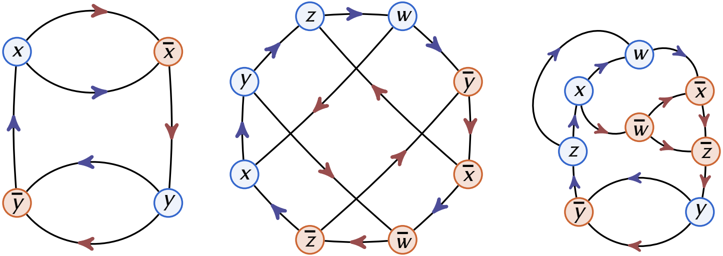
Question. Where does $\frac{1}{16}$ come from?
Question. What is the rôle of the cubic kernels?

Analytic combinatorics ♡ 2-SAT

- A 2-CNF is represented as an *implication digraph.*
- Formula is UNSAT if and only if $\exists x \colon x \rightsquigarrow \bar{x} \rightsquigarrow x$.

Major obstacle. No “tree-unicycle”-style decomposition available. Contradictory components are not disconnected.

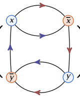
First idea. Inclusion-exclusion using *contradictory patterns*



Understanding $\frac{1}{16}$ and cubic kernels of 2-SAT

Theorem (D. '2019)

As $m = cn$, $c = 1 + \mu n^{-1/3}$, $\mu \to -\infty$ slowly enough, then cubic contradictory kernel of excess r appear with probability $C_r|\mu|^{-3r}$.

- *C_r is equal to the sum $\sum_M 2^{-r}\varkappa(M)/(2r)!$ taken over all possible labelled cubic contradictory components of excess r;*
- $\varkappa\Big($ 

$\Big) = \frac{1}{2} \cdot \frac{1}{2}$ *because of two double edges $x \to \bar{x}$, $y \to \bar{y}$.*

Corollary

$$\mathbb{P}(F_{n,m} \text{ is SAT}) \sim 1 - \frac{1}{16|\mu|^3}$$

Open problem. $\mathbb{P}(F_{n,m}$ is SAT$) =?$ when $\mu \in \mathbb{R}$

A systematic approach to inclusion-exclusion

The philosophy of Analytic Combinatorics

Analytic Combinatorics = Symbolic Method + Asymptotic Analysis

The philosophy of the symbolic method. (Bergeron, Labelle, Leroux).

Combinatorial decomposition $\Rightarrow$ Functional equation

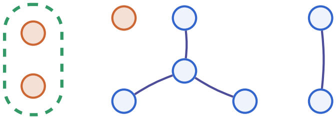
Follow-up: asymptotic analysis. (Flajolet, Odlyzko, ...).

Equation $\Rightarrow$ GF expansion $\Rightarrow$ Asymptotics

Classical Inclusion-Exclusion principle

Additional variables mark special vertices or groups of vertices

$$A(z, w, u) = \sum_{n,k,r} a_{n,k,r} w^k u^r \frac{z^n}{n!}$$



Example:

$a_{n,k,r} = \#_{\text{of}}$ graphs with

- n vertices
- k edges
- r isolated vertices

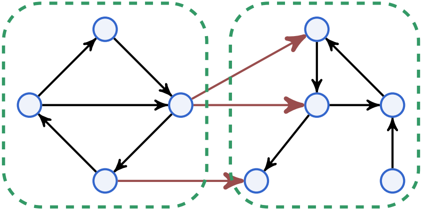
$$B(z, w, u) := A(z, w, u + 1) = \sum_{n,k,r} b_{n,k,r} w^k u^r \frac{z^n}{n!}$$

$b_{n,k,r} = \#_{\text{of}}$ graphs with

- n vertices
- k edges
- r *distinguished* isolated vertices

Graphic convolution rule

$$A(z) = a_0 + \frac{a_1 z}{1! 2^{\binom{1}{2}}} + \frac{a_2 z^2}{2! 2^{\binom{2}{2}}} + \ldots, \quad B(z) = b_0 + \frac{b_1 z}{1! 2^{\binom{1}{2}}} + \frac{b_2 z^2}{2! 2^{\binom{2}{2}}} + \ldots$$



coefficient level	**graphic GF level**
$c_n := \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} 2^{k(n-k)}$	$C(z) = A(z) \cdot B(z)$

Digraph enumeration theorem and its applications

Enumerating digraphs with given SCC

Theorem (Robinson '1973; De Panafieu, D. '2019)

Graphic GF for ***digraphs*** *with strongly connected components from given family SCC is given by*

$$\widehat{D}(z, w) = \frac{1}{e^{-SCC(z,w)} \odot MG(z, -w)}$$

where

- $MG(z, w)$ *is the EGF of multigraphs;*
- $\odot$ *is the exponential Hadamard product*

$$\sum_{n \geqslant 0} a_n \frac{z^n}{n!} \odot \sum_{n \geqslant 0} b_n \frac{z^n}{n!} := \sum_{n \geqslant 0} a_n b_n \frac{z^n}{n!}$$

- *Graphic GF is defined by*

$$\widehat{F}(z, w) = \sum_{n \geqslant 0} f_n(w) \frac{z^n}{e^{\frac{n^2}{2} w} n!}$$

Two main applications of enumeration theorem

Corollary

Graphic GF for acyclic digraphs *is*

$$\widehat{DAG}(z, w) = \frac{1}{MG(-z, -w)}$$

Corollary

Graphic GF for elementary digraphs *(strong components are only isolated vertices and cycles) is*

$$\widehat{S}(z, w) = \frac{1}{MG(-z, -w) + zw\partial_z MG(-z, -w)}$$

Asymptotics of directed acyclic graphs

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Historical overview:

- When $m = cn^2$, [Bender, Richmond, Robinson, Wormald '1984]

Theorem ([Ralaivaosaona, Rasendrahasina, Wagner '20+], [DP., D. '20+])

As $m = cn$, $c < 1$, $n \to \infty$

$$\mathbb{P}(\textit{digraph is acyclic}) \to 1 - c$$

Question. What happens when $c = 1$?

Transition window. $\mu \in \mathbb{R}$.
Answered by Ralaivaosaona, Rasendrahasina, and Wagner: see AofA'2020!

Subcritical phase of DAG enumeration

Theorem (De Panafieu, D. FPSAC'2020)

As $m = n(1 + \mu n^{-1/3})$, $\mu \to -\infty$ sufficiently slowly,

$$\mathbb{P}(D_{n,m} \text{ is acyclic}) \sim |\mu| n^{-1/3} \cdot \sum_{r \geqslant 0} \frac{C_r}{|\mu|^{3r}}$$

Proof.

- Graphic GF for *acyclic digraphs* is $\dfrac{1}{MG(-z,-w)}$
- Write $MG(z, w)$ in the product form (trees, unicycles and complex components)
- Convert graphic GF into EGF + additional combinatorial magic

□

Critical phase of DAG enumeration

Theorem (De Panafieu, D. FPSAC'2020)

As $m = n(1 + \mu n^{-1/3})$, $\mu \in \mathbb{R}$,

$$\#DAG_{n,m} \sim \frac{n!^2 m!}{(2n-m)!} \frac{e^{2n}}{4} \left(\frac{3}{n}\right)^{4/3} \sqrt{\frac{3}{2\pi}} e^{\mu^3/3} \left(H(y) \odot_{y=\frac{1}{3}} \frac{1}{E(y)} \right),$$

where

- $E(z) := \sum_{r \geqslant 0} \frac{(6r)!}{(2r)!(3r)! 2^{5r} 3^{2r}} z^r$
- $H(z) := \sum_{r \geqslant 0} G\left(\frac{3}{2}, \frac{3r}{2} - \frac{1}{4}, -\frac{3^{2/3}}{2}\mu\right) z^r,$
- $G(\lambda, \alpha, x) = \frac{1}{\lambda} \sum_{k \geqslant 0} \frac{(-x)^k}{k!} \frac{1}{\Gamma\left(\frac{\lambda+\alpha-k-1}{\lambda}\right)},$ $\leftarrow$ *"Airy function"*
- $\left(\sum a_n \frac{z^n}{n!}\right) \odot_{z=x} \left(\sum b_n \frac{z^n}{n!}\right) := \sum a_n b_n \frac{x^n}{n!}.$

Critical phase of DAG enumeration

As $m = n(1 + \mu n^{-1/3})$, $\mu \in \mathbb{R}$,

$$\#\mathrm{DAG}_{n,m} \sim \frac{n!^2 m!}{(2n-m)!} \frac{e^{2n}}{4} \left(\frac{3}{n}\right)^{4/3} \sqrt{\frac{3}{2\pi}} e^{\mu^3/3} \left(H(y) \odot_{y=\frac{1}{3}} \frac{1}{E(y)} \right),$$

Theorem (Ralaivaosaona, Rasendrahasina, Wagner '2020)
In the $D_{n,p}$ model, the Hadamard product $\left(H(y) \odot_{y=\frac{1}{3}} \frac{1}{E(y)} \right)$ *can be* replaced by *an integral of the reciprocal Airy function*

$$\frac{C}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-\mu s}}{\mathrm{Ai}(-2^{1/3} s)} ds$$

Phase transition in directed graphs

Directed graphs

Historical overview:

- Strong components are only isolated vertices and cycles below $m = n(1 + \mu n^{-1/3})$ [Łuczak, Seierstad '09]
- Strong components have cubic kernels (for $\mu \in \mathbb{R}$) [Goldschmidt, Stephenson '19]

Theorem (De Panafieu, D. '2020)

As $m = n(1 + \mu n^{-1/3})$, $\mu \to -\infty$,

$$\mathbb{P}\left(\substack{\textit{strong components of } D_{n,m} \\ \textit{are isolated vertices and cycles}} \right) \sim 1 - \frac{1}{2|\mu|^3}.$$

Proof.

- Transform the graphic GF of subcritical digraphs
- Repeat the idea of the previous proof

□

Critical phase of digraph enumeration

Definition. *Elementary digraph* contains only cycles and isolated vertices as strong components.

Theorem (De Panafieu, D. '2020+)

As $m = n(1 + \mu n^{-1/3})$, $\mu \in \mathbb{R}$,

$$\mathbb{P}(D_{n,m} \text{ is elementary}) \sim e^{-\mu^3/6} \sqrt{\frac{3\pi}{2}} \left(H(y) \odot_{y=\frac{1}{3}} \frac{1}{\frac{y}{2} + E(y) + 3y^2 E'(y)} \right),$$

where

- $E(z) := \sum_{r \geqslant 0} \frac{(6r)!}{(2r)!(3r)! 2^{5r} 3^{2r}} z^r$
- $H(z) := \sum_{r \geqslant 0} G\left(\frac{3}{2}, \frac{3r}{2} - \frac{1}{4}, -\frac{3^{2/3}}{2}\mu\right) z^r$,
- $G(\lambda, \alpha, x) = \frac{1}{\lambda} \sum_{k \geqslant 0} \frac{(-x)^k}{k!} \frac{1}{\Gamma\left(\frac{\lambda + \alpha - k - 1}{\lambda}\right)}$. $\leftarrow$ *"Airy function"*

Conclusion

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1. Cubic kernels and their (rational) “compensation factors” play central rôle in the phase transitions of
   - graphs (with or w/o degree constraints)
   - 2-SAT
   - digraphs and acyclic digraphs
2. Most easily seen as expansions in powers of $|\mu|^{-3}$ for the subcritical phase $\mu \to -\infty$.
3. Transition curves when $\mu \in \mathbb{R}$. (in progress for 2-SAT)

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Thank you.