The birth of the strong components

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(an upcoming work)

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First cycles ’1988

The First Cycles in an Evolving Graph

PHILIPPE FLAJOLET, DONALD E. KNUTH AND BORIS PITTEL

The purpose of this paper is to introduce analytical methods by which such questions can be answered systematically. In particular, we will apply the ideas to an interesting question posed by Paul Erdős and communicated by Edgar Palmer to the 1985 Seminar on Random Graphs in Posnań: “What is the expected length of the first cycle in an evolving graph?” The answer turns out to be rather surprising: The first cycle has length $Kn^{1/6} + O(n^{1/8})$ on the average, where

$$K = \frac{1}{\sqrt{8\pi}\, i} \int_{-\infty}^{\infty} \int_{1-i\infty}^{1+i\infty} e^{(\mu+2s)(\mu-s)^2/6} \, \frac{ds}{s} \, d\mu \approx 2.0337 .$$

The form of this result suggests that the expected behavior may be quite difficult to derive using techniques that do not use contour integration.

Directed graphs '2020+

[de Panafieu, D., Ralaivaosaona, Rasendrahasina, Wagner]

Consider a random digraph from $\mathbb{D}(n, p)$ with n vertices, where each edge is drawn independently with probability p and is assigned a random direction (**Gilbert's model**).

- What is the probability that a digraph $\mathbb{D}(n, \frac{1}{n})$ is acyclic?

$$(2n)^{-1/3} e^{3/2} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{\mathrm{Ai}(-2^{1/3} s)} \mathrm{d}s \approx n^{-1/3} \cdot 2.19037\ldots$$

- What is the probability that the strongly connected components of a random digraph $\mathbb{D}(n, \frac{1}{n})$ are isolated vertices or cycles?

$$-2^{-2/3} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{\mathrm{Ai}'(-2^{1/3} s)} \mathrm{d}s \approx 0.69968786651 + \mathcal{O}(n^{-1/3})$$

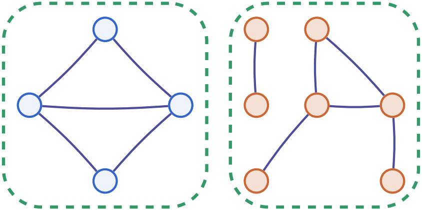
- What is the probability that there is one bicyclic strong complex component in $\mathbb{D}(n, \frac{1}{n})$?

$$\frac{1}{8} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{Ai}(-2; s)}{\mathrm{Ai}'(s)^2} \mathrm{d}s = \frac{1}{8} + \mathcal{O}(n^{-1/3})$$

Part I. Back to the origin: generating functions

The cartesian product

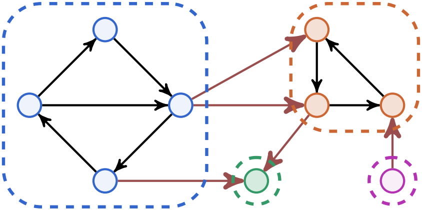
$$\left(a_0 + a_1\frac{z}{1!} + a_2\frac{z^2}{2!} + \dots\right)\left(b_0 + b_1\frac{z}{1!} + b_2\frac{z^2}{2!} + \dots\right) = c_0 + c_1\frac{z}{1!} + c_2\frac{z^2}{2!} + \dots$$

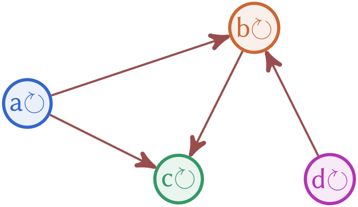


The convolution rule corresponding to EGF:

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

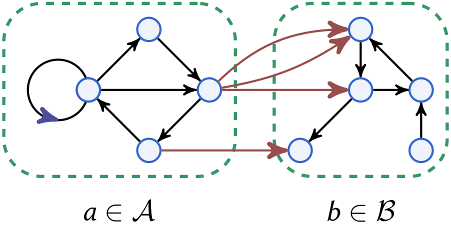
Directed graphs and their components





- Components a↻, b↻, c↻ and d↻ are *strongly-connected* components.
- Components a↻ and d↻ are *source-like* components
- Component c↻ is a *sink-like* component

The arrow product



The graphic generating function (GGF)

Let $\mathcal{F}$ be a family of digraphs and $D \in \mathcal{F}$. Let $n(D)$ denote the number of vertices, and $m(D)$ the number of edges of D.

Their EGF $F(z, w)$ and GGF $\widehat{F}(z, w)$ are defined as

$$F(z,w) := \sum_{D \in \mathcal{F}} \frac{z^{n(D)}}{n(D)!} \frac{w^{m(D)}}{m(D)!}, \quad \widehat{F}(z,w) := \sum_{D \in \mathcal{F}} e^{-n(D)^2 w/2} \frac{z^{n(D)}}{n(D)!} \frac{w^{m(D)}}{m(D)!}$$

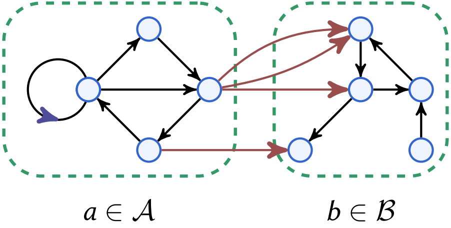
Proposition.

- $\widehat{F}(z,w) = \dfrac{1}{\sqrt{2\pi w}} \displaystyle\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2w}\right) F(ze^{-ix}, w) \mathrm{d}x.$
- Gilbert's random model $\mathbb{P}_{n,p}$ is equidistributed to a Boltzmann distribution with parameter $\lambda = pn$ on the set of all multidigraphs.

$$\mathbb{P}_{n,p}(D \in \mathcal{F}) = e^{-pn^2/2} n! [z^n] \widehat{F}(z, p)$$

The graphic convolution product

$$\left(\sum_{n\geqslant 0} a_n(w) e^{-n^2 w/2} \frac{z^n}{n!}\right) \left(\sum_{n\geqslant 0} b_n(w) e^{-n^2 w/2} \frac{z^n}{n!}\right) = \sum_{n\geqslant 0} c_n(w) e^{-n^2 w/2} \frac{z^n}{n!}$$



The convolution rule corresponding to GGF:

$$c_n(w) = \sum_{k+\ell=n} \binom{n}{k} (e^w)^{k\ell} a_k(w) b_\ell(w).$$

Part II. Families of directed graphs and their generating functions

The main enumeration theorem

Let $\mathcal{S}$ be a family of strongly connected digraphs, and let $\mathcal{D}_{\mathcal{S}}$ be the family of digraphs whose components are constrained to $\mathcal{S}$.

Theorem. GGF of $\mathcal{D}_{\mathcal{S}}$ is given by

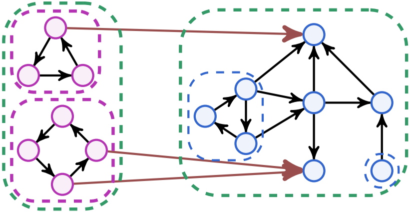
$$\widehat{D}_{\mathcal{S}}(z, w) = \frac{1}{\frac{1}{\sqrt{2\pi w}} \int\limits_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2w} - S(ze^{-ix}, w)\right) \mathrm{d}x}.$$

Moreover, if u marks the source-like components, then

$$\widehat{D}_{\mathcal{S}}(z, w, u) = \frac{\frac{1}{\sqrt{2\pi w}} \int\limits_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2w} + (u-1)S(ze^{-ix}, w)\right) \mathrm{d}x}{\frac{1}{\sqrt{2\pi w}} \int\limits_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2w} - S(ze^{-ix}, w)\right) \mathrm{d}x}.$$

Proof of the main enumeration theorem

Proof. $\mathcal{D}_{\mathcal{S}}$ with distinguished source-like components is an arrow product of a set of strong components and $\mathcal{D}_{\mathcal{S}}$.



 are from distinguished source-like components.

 are from the usual source-like components.

- $\widehat{D}_{\mathcal{S}}(z, w, u+1) = \widehat{e^{uS(z,w)}} \cdot \widehat{D}_{\mathcal{S}}(z, w, 1)$.
- By letting $u = -1$, we obtain $\widehat{D}_{\mathcal{S}}(z, w) = \dfrac{1}{\widehat{e^{-S(z,w)}}}$.
- By plugging $u \mapsto u - 1$, we obtain $\widehat{D}_{\mathcal{S}}(z, w, u) = \dfrac{\widehat{e^{(u-1)S(z,w)}}}{\widehat{e^{-S(z,w)}}}$.

DAGs and elementary digraphs

Application 1. In DAGs, the only possible strong components are isolated vertices, $S(z,w) = z$.

$$\widehat{D}_{DAG}(z,w) = \frac{1}{\frac{1}{\sqrt{2\pi w}} \int\limits_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2w} - ze^{-ix}\right) \mathrm{d}x}.$$

Application 2. The *elementary digraphs* are those whose strong components are isolated vertices or cycles, $S(z,w) = z + \ln \frac{1}{1-zw}$.

$$\widehat{D}_{elem}(z,w) = \frac{1}{\frac{1}{\sqrt{2\pi w}} \int\limits_{-\infty}^{\infty} \frac{1 - zwe^{-ix}}{\exp\left(\frac{x^2}{2w} + ze^{-ix}\right)} \mathrm{d}x}.$$

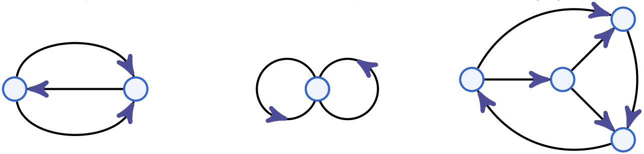
Complex components

The Birth of the Giant Component

Dedicated to Paul Erdős on his 80th birthday

Svante Janson, Donald E. Knuth, Tomasz Łuczak, and Boris Pittel

Is there a simple recurrence governing the leading coefficients $s_{10}, s_{20}, s_{30}, \ldots$, perhaps analogous to the relation we observed for ordinary connected components in (8.5)?



The EGF of strong components of *excess* r is

$$\mathrm{Strong}_r(z, w) = s_r w^r \frac{(zw)^{2r}}{(1 - zw)^{3r}} + w^r \frac{Q_r(zw)}{(1 - zw)^{3r-1}}.$$

$$(s_r)_{r=1}^{\infty} = \left(\frac{1}{2}, \frac{17}{8}, \frac{275}{12}, \frac{26141}{64}, \frac{1630711}{160}, \ldots \right).$$

Elementary digraphs with one bicyclic component

Application 3. Let $\widehat{H}_{\mathrm{bicycle}}$ be the GGF of elementary digraphs with one bicyclic component. Then,

$$\widehat{H}_{\mathrm{bicycle}}(z,w) \sim \frac{\frac{1}{\sqrt{2\pi w}} \int_{-\infty}^{\infty} \frac{1}{2} \frac{w^3 z^2 e^{-2ix}}{(1 - zwe^{-ix})^2} e^{-\frac{x^2}{2w} - ze^{-ix}} \mathrm{d}x}{\left(\frac{1}{\sqrt{2\pi w}} \int_{-\infty}^{\infty} (1 - zwe^{-ix}) e^{-\frac{x^2}{2w} - ze^{-ix}} \mathrm{d}x \right)^2}.$$

Proof. Apply the enumeration theorem with

$$S(z,w,\mathbf{v}) := z + \ln \frac{1}{1 - zw} + \mathbf{v} \cdot S_{\mathrm{bicycle}}(z,w),$$

where

$$S_{\mathrm{bicycle}}(z,w) = \frac{1}{2} \left(\frac{w^3 z^2}{(1 - zw)^3} + \frac{w^2 z}{(1 - zw)^2} \right)$$

and extract $[\mathbf{v}^1]$.

Source-like complex component

Generalised enumeration theorem. Let $\mathcal{S}$ and $\mathcal{H}$ be two disjoint families of strongly connected digraphs, and let $\mathcal{D}_{\mathcal{S},\mathcal{H}}$ be the family of digraphs whose components are contrained to $\mathcal{S}$ and $\mathcal{H}$. Let $\mathbf{u}$ and $\mathbf{v}$ mark source-like components from $\mathcal{S}$ and $\mathcal{H}$. Then,

$$\widehat{D_{\mathcal{S},\mathcal{H}}}(z, w, \mathbf{u}, \mathbf{v}) = \frac{\displaystyle\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2w} + (\mathbf{u}-1)S(ze^{-ix}, w) + (\mathbf{v}-1)H(ze^{-ix}, w)\right) \mathrm{d}x}{\displaystyle\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2w} - S(ze^{-ix}, w) - H(ze^{-ix}, w)\right) \mathrm{d}x}$$

Application 4. GGF of elementary digraphs with one source-like complex component from $\mathcal{S}$ is

$$W_{\mathcal{S}}(z, w) = \frac{\displaystyle\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2w}\right) S(ze^{-ix}, w) \mathrm{d}x}{\displaystyle\int_{-\infty}^{+\infty} (1 - zwe^{-ix}) \exp\left(-\frac{x^2}{2w} - ze^{-ix} - S(ze^{-ix}, w)\right) \mathrm{d}x}$$

Proof. Take $H(z, w) = z + \ln \frac{1}{1-zw}$. Put $\mathbf{v} = 1$ and extract $[\mathbf{u}^1]$.

Part III. Asymptotic analysis

Asymptotic analysis: general scheme

- The probabilities of interest can be expressed as

$$\mathbb{P}_{n,p}(D \in \mathcal{F}) = e^{-pn^2/2} n! [z^n] \widehat{F}(z,p).$$

- $[z^n]\widehat{F}(z,p) = \dfrac{1}{2\pi i} \displaystyle\oint_{|z|=R} \dfrac{\widehat{F}(z,p)}{z^{n+1}} \mathrm{d}z.$
- For a given value of $p \to 0^+$, and for z fixed, find an asymptotic approximation of $\widehat{F}(z,p)$.
  - $\widehat{F}(z,p)$ is a product of integrals itself, each integral over $\mathbb{R}$.
  - Change the contour: preserve the starting and the finishing points, but let it pass through $x = x_0 \in i\mathbb{R}$ in the middle.
  - The dominant contribution is around $x = x_0 + \varepsilon$.

Horizontal Path. Path Γ.

- Dominant part of $[z^n]\widehat{F}(z,p)$ is when z is around $R \pm 0i$.

Asymptotics of the deformed exponent

Let $T(zw)$ and $U(zw)$ be the EGF of rooted and unrooted trees.

$$\phi(z, w) := \frac{1}{\sqrt{2\pi w}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2w} - z e^{-ix}\right) \mathrm{d}x = \sum_{n \geqslant 0} e^{-n^2 w/2} \frac{(-z)^n}{n!}.$$

(a) If $zw \in [0, e^{-1})$, then $\phi(z, w) \sim \dfrac{e^{-U(zw)/w}}{\sqrt{1 - T(zw)}}$

(b) If $1 - ezw = \theta w^{2/3}$, $\theta \to \infty$, then

$$\phi(z, w) \sim (2\theta)^{-1/4} w^{-1/6} \exp\left(-\frac{1}{2w} + \frac{\theta}{w^{1/3}} - \frac{2^{3/2}}{3}\theta^{3/2}\right)$$

(c) If $1 - ezw = \theta w^{2/3}$, $\theta \in \mathbb{C}$, then

$$\phi(z, w) \sim 2^{5/6} \pi^{1/2} w^{-1/6} \mathrm{Ai}(2^{1/3}\theta) \exp\left(-\frac{1}{2w} + \frac{\theta}{w^{1/3}}\right).$$

Generalised Airy function

The Airy function satisfies a linear differential equation

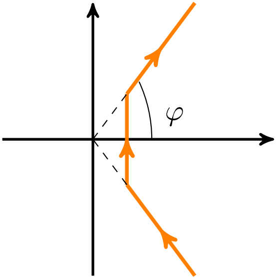
$$\mathrm{Ai}(z)'' - z\mathrm{Ai}(z) = 0$$

It can be expressed as an integral and its derivatives as well

$$\partial_z^r \mathrm{Ai}(z) = \frac{(-1)^r}{2\pi i} \int_{-i\infty}^{+i\infty} t^r \exp(-zt + t^3/3) \mathrm{d}t.$$

It is natural to extend this definition, so that $r \in \mathbb{Z}$ and deform the contour a little bit:

$$\mathrm{Ai}(r; z) := \frac{(-1)^r}{2\pi i} \int_{t \in \Pi(\varphi)} t^r \exp(-zt + t^3/3) \mathrm{d}t.$$



Generalised deformed exponent

Let $T(zw)$ and $U(zw)$ be the EGF of rooted and unrooted trees.

$$\psi_r(z, w) := \frac{1}{\sqrt{2\pi w}} \int_{-\infty}^{+\infty} (1 - zwe^{-ix})^r \exp\left(-\frac{x^2}{2w} - ze^{-ix}\right) \mathrm{d}x.$$

(a) If $zw \in [0, e^{-1})$, then $\psi_r(z, w) \sim e^{-U(zw)/w}(1 - T(zw))^{r-1/2}$

(b) If $1 - ezw = \theta w^{2/3}$, $\theta \to \infty$, then

$$\psi_r(z, w) \sim (2\theta)^{r/2-1/4} w^{r/3-1/6} \exp\left(-\frac{1}{2w} + \frac{\theta}{w^{1/3}} - \frac{2^{3/2}}{3}\theta^{3/2}\right)$$

(c) If $1 - ezw = \theta w^{2/3}$, $\theta \in \mathbb{C}$, then

$$\psi_r(z, w) \sim C \cdot D^r \cdot w^{-1/6+r/3} \mathrm{Ai}(r; 2^{1/3}\theta) \exp\left(-\frac{1}{2w} + \frac{\theta}{w^{1/3}}\right).$$

Computing the asymptotic probabilities

Theorem. In the multidigraph model, when $p = \frac{1}{n}(1 + \mu n^{-1/3})$,

- $\mathbb{P}_{n,p}(D_{n,p} \text{ is acyclic}) \sim (2n)^{-1/3} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-\mu s - \mu^3/6}}{\mathrm{Ai}(-2^{1/3}s)} \mathrm{d}s$
- $\mathbb{P}_{n,p}(D_{n,p} \text{ is elementary}) \sim -2^{-2/3} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-\mu s - \mu^3/6}}{\mathrm{Ai}'(-2^{1/3}s)} \mathrm{d}s$

The probability to have one complex component of excess r is asymptotically equal to

$$\mathbb{P}_{n,p}(\cdot) \sim s_r \cdot C \cdot D^r \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{Ai}(1 - 3r; -2^{1/3}s)}{(\mathrm{Ai}'(-2^{1/3}s))^2} e^{-\mu s - \mu^3/6} \mathrm{d}s.$$

Outside the critical window

- When $p = \lambda n^{-1}$, $\lambda < 1$, the probabilities can be obtained by applying large powers theorem to

$$\psi_r(z, w) \sim e^{-U(zw)/w}(1 - T(zw))^{r-1/2}$$

  for $zw < e^{-1}$.
- When $p = \lambda n^{-1}$, $\lambda > 1$, the knowledge of the roots of $\psi_r(z, w)$ is sufficient.
- When $p = n^{-1}(1 + \mu n^{-1/3})$, and $\mu \to -\infty$, we can apply semi-large powers theorem.

$$\mathbb{P}_{n,p}(D_{n,p} \text{ is elementary}) \sim 1 - \frac{1}{2|\mu|^3} + \mathcal{O}(|\mu|^{-6})$$

Part IV. Multidigraphs and simple digraphs.

Simple digraphs and multidigraphs

The basic deformed exponent corresponding to simple digraphs is

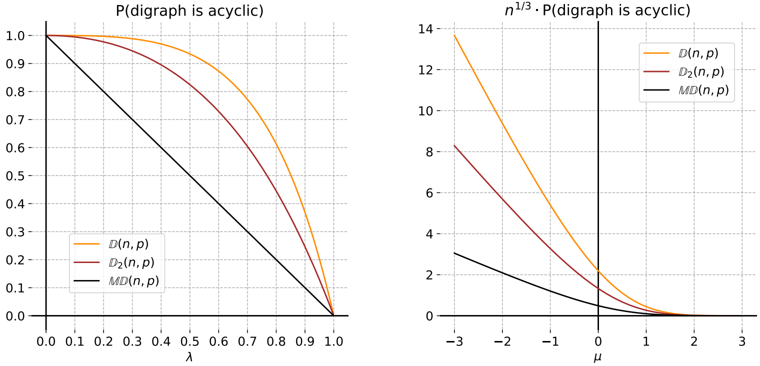
$$\phi^{(simple)}(z, w) = \phi(z\sqrt{1+w}, \log(1+w)).$$

Elementary digraphs can be adjusted by forbidding loops and 2-cycles which yields

$$S(z, w) = z + \ln \frac{1}{1 - zw} - zw - \frac{(zw)^2}{2}.$$

All the obtained asymptotic approximations can be readily used to obtain the asymptotics of simple digraphs directly.

Simple digraphs and multidigraphs



Part V. Numerical results

Empirical results for different families

Table 3: The rescaled probability $n^{1/3}\mathbb{P}(n, \frac{1}{n}(1+\mu n^{-1/3}))$ that a random multidigraph is acyclic, for $\mu \in \{-3,-2,-1,0,1,2,3\}$.

n	$\mu=-3$	$\mu=-2$	$\mu=-1$	$\mu=0$	$\mu=1$	$\mu=2$	$\mu=3$
100	3.00670	2.03423	1.14068	0.46304	0.11671	0.01642	0.00124
500	3.02011	2.05620	1.16642	0.47393	0.11007	0.01147	0.00044
1000	3.02522	2.06335	1.17410	0.47705	0.10793	0.01012	0.00030
2000	3.02963	2.06925	1.18025	0.47950	0.10615	0.00908	0.00021
3000	3.03190	2.07219	1.18326	0.48068	0.10527	0.00859	0.00018
∞	3.04943	2.09362	1.20431	0.48873	0.09876	0.00550	0.00004

Table 4: The probability $\mathbb{P}(n, \frac{1}{n}(1+\mu n^{-1/3}))$ that a random multidigraph is elementary, for $\mu \in \{-3,-2,-1,0,1,2,3\}$.

n	$\mu=-3$	$\mu=-2$	$\mu=-1$	$\mu=0$	$\mu=1$	$\mu=2$	$\mu=3$
100	0.99782	0.98513	0.92149	0.71322	0.36692	0.10522	0.01574
500	0.99368	0.97673	0.91099	0.70807	0.35174	0.08151	0.00698
1000	0.99216	0.97411	0.90794	0.70645	0.34684	0.07442	0.00511
2000	0.99086	0.97200	0.90553	0.70512	0.34279	0.06878	0.00386
3000	0.99020	0.97095	0.90435	0.70446	0.34077	0.06604	0.00333
∞	0.98521	0.96354	0.89622	0.69968	0.32582	0.04740	0.00089

Empirical results for different families

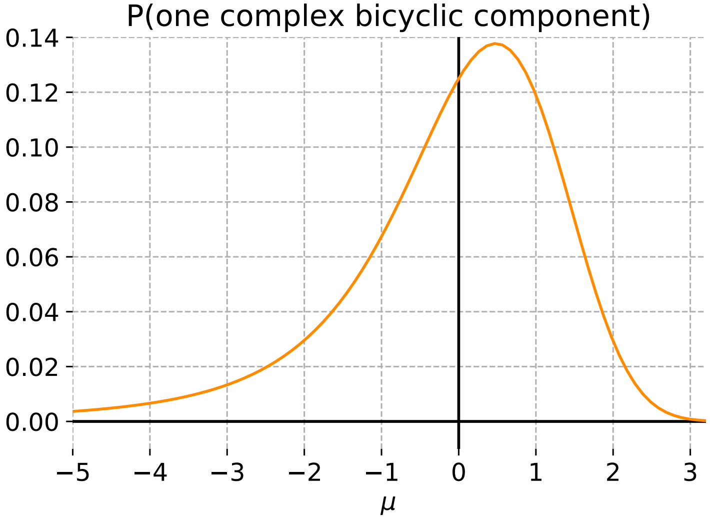
Table 5: The probability $\mathbb{P}(n, \frac{1}{n}(1+\mu n^{-1/3}))$ that a random multidigraph has one bicyclic complex component with a cubic kernel, for $\mu \in \{-3, -2, -1, 0, 1, 2, 3\}$.

n	$\mu = -3$	$\mu = -2$	$\mu = -1$	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
100	0.00626	0.02655	0.09257	0.19929	0.20576	0.09385	0.01942
500	0.01007	0.02936	0.08213	0.16448	0.16422	0.06160	0.00740
1000	0.01095	0.02972	0.07905	0.15549	0.15372	0.05377	0.00521
2000	0.01158	0.02988	0.07661	0.14867	0.14584	0.04801	0.00382
3000	0.01185	0.02992	0.07543	0.14546	0.14215	0.04535	0.00325
∞	0.01333	0.02958	0.06737	0.12500	0.11896	0.02931	0.00081

Table 6: The probability $\mathbb{P}(n, \frac{1}{n})$ that a random (multi-)digraph is elementary.

n	$\mathbb{D}(n, \frac{1}{n})$	$\mathbb{D}_2(n, \frac{1}{n})$	$\mathbb{MD}(n, \frac{1}{n})$
100	0.77731	0.74346	0.71322
200	0.75520	0.73091	0.71074
500	0.73472	0.71969	0.70807
1000	0.72424	0.71403	0.70645
2000	0.71686	0.71003	0.70512
∞	0.69968	0.69968	0.69968

One bicyclic component



Bonus. A mysterious connection

Maximum of Brownian motion with parabolic drift

[Knessl'2000],[Janson,Louchard,Martin-Löf'2010],[Janson'2012]

A special function

$$Q(x,\mu) := e^{\mu x} 2^{2/3} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\mu\tau - \mu^3/6} \frac{\mathrm{Ai}(2^{1/3}(x+\tau))}{\mathrm{Ai}(2^{1/3}\tau)^2} \mathrm{d}\tau$$

arises as a solution to the system of partial differential equations

$$\begin{cases} Q_\mu = \frac{1}{2} Q_{xx} - \mu Q_x, & -\infty < \mu < \infty,\ x > 0; \\ \frac{1}{2} Q_x(0,\mu) - \mu Q(0,\mu) = 0, & -\infty < \mu < \infty; \\ Q \to \delta(x), & \mu \to -\infty. \end{cases}$$

- When $x = 0$, this corresponds to critical DAGs.
- What corresponds to Taylor expansion at $x = 0$?

Conclusion

Conclusion

1. The phase transition curves for DAG, elementary digraphs and analysis of complex components is finally completed. The technique encompasses different digraph models (with or without loops or 2-cycles) and gives precise results for them.
2. Still a lot of questions open (and probably doable!):
   - Statistics of random DAGs (sinks, sources)
   - Asymptotics of strongly connected graphs
   - Simultaneous asymptotics of sink-like and source-like components
   - Cubic kernels (digraphs)
   - Digraphs with degree contraints
   - Giant component of a digraph
   - Triple, quadruple arrow product?
   - Analysis of 2-SAT with similar level of precision
   - …(enough for a PhD thesis or more) …
3. Mysterious connection with maxima of Brownian motions?

Thank you for your attention.