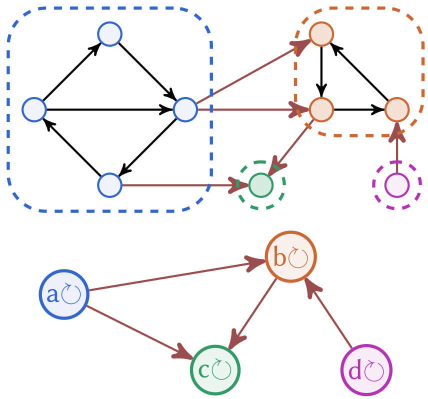
Counting directed acyclic and elementary digraphs (by Élie de Panafieu and Sergey Dovgal)

Directed graphs and their generating functions



- **Directed acyclic graphs**: stronly connected components are vertices
- **Elementary digraphs**: strongly connected components are vertices and cycles
- $a_{n,m}$:= the number of digraphs with n vertices and m edges from a given family.

- **Exponential generating function**:

$$A(z,w) := \sum_{n,m} a_{n,m} w^m \frac{z^n}{n!}$$

- **Graphic generating function**:

$$\boldsymbol{A}(z,w) := \sum_{n,m} a_{n,m} w^m \frac{z^n}{n!(1+w)^{\binom{n}{2}}}$$

- **Exponential Hadamard product:**

$$\Big(\sum_{n\geqslant 0} a_n \frac{z^n}{n!}\Big) \odot_z \Big(\sum_{n\geqslant 0} b_n \frac{z^n}{n!}\Big) = \sum_{n\geqslant 0} a_n b_n \frac{z^n}{n!}$$

Theorem. If the strong components of $\mathcal{D}$ are restricted to the family $\mathcal{S}$, and $S(z,w)$ is its EGF, then the GGF of $\mathcal{D}$ is

$$\boldsymbol{D}(z,w) = \frac{1}{e^{-S(z,w)} \odot_z \mathbf{Set}(z,w)}$$

$$\mathbf{Set}(z,w) = \sum_{n\geqslant 0} \frac{z^n}{n!(1+w)^{\binom{n}{2}}}$$

1. Generating functions.

- $S_{DAG}(z,w) = z$
- $\boldsymbol{D}_{DAG} = \dfrac{1}{\mathbf{Set}(-z,w)}$
- $S_{elem}(z,w) = z + \ln \dfrac{1}{1-zw} - zw$
- $\boldsymbol{D}_{elem} = \dfrac{1}{\mathbf{Set}(-z,w) + z\frac{w}{1-w}\frac{d}{dz}\mathbf{Set}(-z,w)}$

2. Product representation.

$$\mathbf{Set}(z,w) = G\left(z, \frac{-w}{1+w}\right)$$

- Generating function of **graphs:**

$$G(z,w) = \sum_{n\geqslant 0} \frac{z^n}{n!}(1+w)^{\binom{n}{2}} = e^{U(zw)/w + V(zw)} \sum_{k\geqslant 0} \mathrm{Complex}_k(zw) w^k$$

- U, T, V and $\mathrm{Complex}_k$ are EGF of trees, rooted trees, unicycles and components of excess k

3. The architecture of the complex component in graphs.

- $\text{Complex}_r(z) \sim e_r \dfrac{T(z)^{5r}}{(1-T(z))^{3r}} + \dfrac{P_r(T(z))}{(1-T(z))^{3r-1}}$,
- $e_r = \dfrac{(6r)!}{2^{5r}3^{2r}(2r)!(3r)!}$
- $E(y) := \sum_{r\geqslant 0} e_r y^r, \quad e_r^{(-1)} = [y^r]\dfrac{1}{E(-y)}$

Theorem.

(a) If $m = \lambda n$, and $\lambda < 1$, then a random digraph is acyclic with probability

$$\mathbb{P}(n,m) \sim e^{\lambda}(1-\lambda)$$

(b) If $m = n(1+\mu n^{-1/3})$, and μ stays in a bounded real interval, or $\mu \to -\infty$, then a random digraph is acyclic with probability

$$\mathbb{P}(n,m) \sim \sqrt{2\pi} n^{-1/3} \frac{3^{5/6}}{2} e^{1-\mu^3/6} \sum_{r\geqslant 0} 3^{-r} e_r^{(-1)} s_r^-(\mu)$$

4. Two special generating functions.

- $S^+(y,\mu) := \sum_{r\geqslant 0} H(\frac{3r}{2}+\frac{1}{4}, -\frac{3^{2/3}}{2}\mu) y^r = \sum_{r\geqslant 0} s_r^+(\mu)$
- $S^-(y,\mu) := \sum_{r\geqslant 0} H(\frac{3r}{2}-\frac{1}{4}, -\frac{3^{2/3}}{2}\mu) y^r = \sum_{r\geqslant 0} s_r^-(\mu)$
- $H(r,x) := \dfrac{2}{3} \sum_{k\geqslant 0} \frac{1}{\Gamma(\frac{2r-2k+1}{3})} \frac{(-x)^k}{k!}$

Theorem.

(a) If $m = n(1+\mu n^{-1/3})$, and $\mu \to -\infty$, then a random digraph is elementary with probability

$$\mathbb{P}(n,m) \sim 1 - \frac{1}{2|\mu|^3}$$

(b) If $m = n(1+\mu n^{-1/3})$, and μ stays in a bounded real interval, or $\mu \to -\infty$, then a random digraph is elementary with probability

$$\mathbb{P}(n,m) \sim e^{-\mu^3/6} \sqrt{\frac{3\pi}{2}} \sum_{r\geqslant 0} 3^{-r} s_r^+(\mu) \cdot [y^r] \frac{1}{y/2 + E(y) + 3y^2 E'(y)}.$$

5. The exact expressions for directed acyclic and elementary digraphs:

$$\#_{DAG}(m,n) = n!^2 \sum_{t\geqslant 0} [z_0^n z_1^n] \frac{(U(z_0)+U(z_1))^{2n-m+t}}{(2n-m+t)!} \frac{e^{U(z_1)+V(z_0)}}{e^{V(z_1)}} [y^t] \frac{\sum_{j\geqslant 0} \mathrm{Complex}_j(z_0) y^j}{\sum_{k\geqslant 0} \mathrm{Complex}_k(z_1)\left(-\frac{y}{1+y}\right)^k} \frac{1}{(1+y)^n};$$

$$\#_{elem}(m,n) = n!^2 \sum_{t\geqslant 0} [z_0^n z_1^n] \frac{(U(z_0)+U(z_1))^{2n-m+t}}{(2n-m+t)!} \frac{e^{V(z_0)}}{e^{V(z_1)}} [y^t] \frac{\exp\left(\frac{2U(z_1)}{1-y}\right)\left(\frac{1-y}{1+y}\right)^n \sum_{j\geqslant 0} \mathrm{Complex}_j(z_0) y^j}{\sum_{k\geqslant 0}\left[\mathrm{Complex}_k(z_1)\left(1-\frac{1+y}{1-y}(T(z_1)-z_1\frac{d}{dz_1}V(z_1))\right)+\frac{1+y}{1-y}z_1\frac{d}{dz_1}\mathrm{Complex}_k(z_1)\right]\left(-y\frac{1-y}{1+y}\right)^k}$$

6. Bivariate semi-large powers lemma.

Lemma. If $n, m \to \infty$ and $m = n(1+\mu n^{-1/3})$ and $F(z_0, z_1)$ is analytic in $\{z_0, z_1 \in \mathbb{C}\colon |z_0| < 1, |z_1| < 1\}$, then

$$[z_0^n z_1^n]\left(U(z_0)+U(z_1)\right)^{2n-m} \frac{F(T(z_0),T(z_1))}{(1-T(z_0))^{r_0}(1-T(z_1))^{r_1}} \sim \frac{e^{2n}}{4}\left(\frac{3}{n}\right)^{(4-r_0-r_1)/3} F\left(\frac{m}{n},\frac{m}{n}\right) H\left(\frac{r_0}{2}, -\frac{3^{2/3}}{2}\mu\right) H\left(\frac{r_1}{2}, -\frac{3^{2/3}}{2}\mu\right)$$

where

$$H(r,x) := \frac{2}{3}\sum_{k\geqslant 0} \frac{1}{\Gamma\left(\frac{2r-2k+1}{3}\right)} \frac{(-x)^k}{k!}$$

7. Conclusions and Acknowledgements.

- This work was started independently of Naina Ralaivaosaona, Vonjy Rasendrahasina and Stephan Wagner, who are solving the problem of enumeration of DAGs but express their answer in a different form;
- We have since joined our efforts to extend the analysis onto a broader class of digraphs

- We are grateful to Olivier Bodini and Vlady Ravelomanana for fruitful discussions and to the anonymous referees whose suggestions helped to improve the paper
- This work is partly supported by ANR project METACONC.