Équations differentialles ordinaires, solutions pour la feuille TD5.

Sergey Dovgal

Introduction au calcul scientifique, SPI L2

1 Cauchy form

Ordinary differential equations in Cauchy form is a very specific formulation of an ordinary differential equation. It is written as follows:

Differential equation in Cauchy form.

$$\begin{cases} y'(t) = f(y(t), t), \quad t \in [a, b]; \\ y(a) = y_0 \end{cases} \tag{1}$$

Example 1. Consider the following differential equation:

$$\begin{cases} y'(t) = -ty^2(t), \quad t \in [0, 1]; \\ y(0) = 2 \end{cases} \tag{2}$$

Here, in order to specify the differential equation explicitly in Cauchy form (1), we should specify

(i) The bivariate function $f(y, t)$ for (1);

(ii) The target interval $[a, b]$;

(iii) The initial value y_0.

In particular, here

(i) $f(y, t) = -t \cdot y^2$;

(ii) $a = 0$, $b = 1$;

(iii) $y_0 = 2$.

Remark 1. Note that y and t are independent arguments of $f(y, t)$ in the sense that you can substiute concrete values, for example $y = 2$, $t = 1$, and evaluate your function at these values:

$$f(2, 1) = -4.$$

Example 2.

$$\begin{cases} y'(t) = 1 + y(t), \quad t \in [0, 1], \\ y(0) = 0 \end{cases} \tag{3}$$

Again, we need to provide $f(y, t)$, initial conditions a, y_0, and the right border b of the interval. They are given by $f(y, t) = 1 + y$, $a = 0$, $y_0 = 0$, and $b = 1$.

Theorem 1 (Cauchy–Lipschitz)**.** Supposons que la fonction $f(t, y)$ est

(i) Continue par rapport à ses deux variables;

(ii) Lipschitzienne par rapport à y de rapport L qualconque

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

pour tout t, y_1, y_2.

Dans ce cas le problème de Cauchy admet une solution unique $y(t)$ et cette solution a deux derivées continues.

Example 3. It is not always so easy to write a differential equation in Cauchy form. Imagine the following strange example:

$$e^{f'(t)} + f(t) \cdot f'(t) = tf(t)$$

Here, we can't express $f'(t)$ as a function explicitly depending on $f(t)$ and t, as this dependence would not be explicit.

2 Family of Euler's methods

In the course there were described three different Euler methods, namely, explicit, implicit and modified. Among theree of them, implicit is the most difficult to implement, because it requires solving an additional supplementary equation. The fourth method, with a different name, but related to the aforementioned Euler methods, is Crank–Nicolson's method, which is a mixture of implicit and explicit Euler's methods.

Explicit Euler	$y_{k+1} = y_k + hf(t_k, y_k)$
Implicit Euler	$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$
Modified Euler	$y_{k+1} = y_k + hf(t_{k+1/2}, y_{k+1/2})$ $y_{k+1/2} = y_k + \frac{h}{2} f(t_k, y_k)$
Crank–Nicolson	$y_{k+1} = y_k + h\dfrac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}$

Table 1: Family of four iterative methods

The target interval $[a, b]$ is divided into N parts, and the sequence t_k is assigned as

$$t_k = a + hk = a + k\frac{b-a}{N},$$

so that $t_0 = a$ and $t_N = b$. The iteration starts recursively with y_0 given, and then we compute all further $y_1, y_2, \ldots$ until y_N.

Algorithm 1 provides pseudocode to Euler's explicit method.

Algorithm 1: Explicit Euler's method

```
Data: number N of parts in which we divide the interval [a, b]
Result: Sequence of values u_0, u_1, ..., u_n
Function Euler(N):
    h := (b − a)/N ;
    for k = 1..N do
        t_k = a + hk ;
        u_{k+1} := u_k + h f(t_k, u_k) ;
    return [u_0, u_1, ..., u_N] ;
```

3 Solutions to exercises

Exercise 1. Soit $y(t)$ une fonction à valeurs positives polution de :

$$\begin{cases} y'(t) + ty^2(t) = 0, & t > 0 \\ y(0) = 2 & . \end{cases}$$

(i) Écrire ce problème sous la forme d'un problème de Cauchy;

(ii) Écrire en pseudo-code la méthode d'Euler explicite pour la résolution de ce problème pour un certain pas de discrétisation h;

(iii) Quelles sont les équations à résoudre lorsqu'on applique le schema d'Euler implicite pour résoudre ce problème ?

Solution. We start by writing the problem in Cauchy form. This is done by putting the differential equation into the form $y'(t) = f(t, y)$:

$$\begin{cases} y'(t) = -ty^2(t), & t > 0 \\ y(0) = 2 & . \end{cases}$$

More specifically, we have $f(t, y) = -ty^2$. One thing is problematic here, namely, a usual Cauchy problem (1) is defined on a bounded interval $t \in [a, b]$, but here we have an unbounded $t \in (0, +\infty)$. So, we need to choose in an arbitrary manner the right tail T of the interval:

$$\begin{cases} y'(t) = -ty^2(t), & t \in [0, T] \\ y(0) = 2 & . \end{cases}$$

By substituting concrete values and functions into Algorithm 1, we obtain the pseudocode for Euler iteration for this particular exercise.

Algorithm 2: Explicit Euler's method for exercise 1

```
Data: number N of parts in which we divide the interval [0, T]
Result: Sequence of values u_0, u_1, ..., u_n
Function Euler(N):
    h := T/N ;
    for k = 0 .. N − 1 do
        t_k = hk ;
        u_{k+1} := u_k − h t_k u_k^2 ;
    return [u_0, u_1, ..., u_N] ;
```

Now, let us switch to the implicit Euler's iteration. The recurrence of this iteration takes the form

$$u_{k+1} = u_k - ht_{k+1}u_{k+1}^2$$

which is a quadratic equation with respect to u_{k+1}. We have two alternative solutions:

$$u'_{k+1} = \frac{-1 + \sqrt{1 + 4u_k ht_{k+1}}}{2ht_{k+1}}, \quad u''_{k+1} = \frac{-1 - \sqrt{1 + 4u_k ht_{k+1}}}{2ht_{k+1}}.$$

Normally, the presence of these two alternatives doesn't directly allow us to pick one variant for the implicit method. This gives rise to the two possibilities. However, *heuristically*, we can choose the branch, whose value u_{k+1} is closer to the previous obtained value u_k. So, among the two branches we will choose the solution with the positive sign, i.e.

$$u_{k+1} = \frac{-1 + \sqrt{1 + 4u_k ht_{k+1}}}{2ht_{k+1}}.$$

See the full description in Algorithm 3.

Algorithm 3: Implicit Euler's method for exercise 1

Data: number N of parts in which we divide the interval $[0, T]$
Result: Sequence of values $u_0, u_1, \ldots, u_n$
Function *Euler*(N)**:**

$h := \dfrac{T}{N}$;
for $k = 0 \ .. \ N-1$ **do**
$t_k = hk$;
$u_{k+1} := \dfrac{-1 + \sqrt{1 + 4u_k ht_{k+1}}}{2ht_{k+1}}$;
return $[u_0, u_1, \ldots, u_N]$;

Finally, we may be curious to find out the explicit form of the solution. Unfortunately, if we try to apply Cauchy–Lipshitz theorem to try to prove the uniqueness of the solution with given initial conditions, we fail. Also, the presence of two solutions in the implicit method indicates that something unusual is hapenning. In fact, it may happen on practice that a differential equation in Cauchy form may have multiple solutions, and the number of the solutions may also depend on the initial conditions.

In the case of differential equation $y'(t) = -ty^2(t)$ we can apply a "physical" heuristics, with no guarantee whatsoever.

Let us start with the differential equation mentioned in the problem statement.

$$\frac{dy}{dt} = -ty^2$$

We imagine that $\dfrac{dy}{dt}$ is actually not a derivative, but a fraction. This allows us to multiply both sides by dt:

$$dy = -ty^2 dt.$$

Next, we separate the variables y and t, so that one side of the equation contains only y and dy, and the second side contains only t and dt:

$$\frac{dy}{y^2} = -tdt$$

Taking the indefinite integral, we get

$$-\frac{1}{y} = -\frac{t^2}{2} - C$$

which implies

$$y(t) = \frac{1}{\frac{t^2}{2} + C}.$$

It is possible to verify that there exists a unique C such that the abovementioned function satisfies the given functional equation ($C = 1/2$). It is remarkable that the described technique allows to find a solution, but doesn't give a guarantee that this is the only solution existing. □

Exercise 2. On considère le problème de Cauchy suivant :

$$\begin{cases} y'(t) = 1 + y(t), & t \in [0,1] \\ y(0) = 0. \end{cases}$$

(i) Montrer que ce problème admet une solution unique. Donner l'expression explicite de cette solution.

(ii) Calculer des valeurs approchées de $y(0.1), y(0.2), \ldots, y(1)$ en utilisant la méthode d'Euler explicite avec $h = 0.1$.

(iii) Tracer la solution exacte et les solutions approchées sur le même graphique.

(iv) Calculer des valeurs approchées de $y(0.1), y(0.2), \ldots, y(1)$ en utilisant la méthode d'Euler modifiée avec $h = 0.1$.

(v) Comparer les erreurs faites avec la méthods d'Euler explicite et avec la méthode d'Euler modifiée.

Solution. In order to find the exact solution, we use the same heuristics as in the previous exercise. We represent the derivative as a fraction

$$\frac{dy}{dt} = 1 + y$$

After, multiply by dt and divide by $1 + y$:

$$\frac{dy}{1+y} = dt$$

By taking the integral on both sides we obtain a solution up to a constant:

$$\ln(1 + y) = t + C$$

so that

$$y(t) = e^{t+C} - 1.$$

The constant C is discovered from the initial condition $y(0) = 0$ and is equal to zero: $C = 0$. Finally, $y(t) = e^t - 1$.

This (heuristic) solution is perfectly fitting the initial differential equation and the initial conditions, but we also need to prove the uniqueness of this solution, using Cauchy–Lipshitz theorem.

In the Cauchy form (1), the function $f(t, y)$ of the current exercise has the form

$$f(t, y) = 1 + y$$

and is continuous as a sum of two continuous functions. This function is Lipsthitz with constant 1 with respect to its second argument:

$$|f(t, y_1) - f(t, y_2)| = 1 \cdot |y_1 - y_2|.$$

This implies the uniqueness of the solution.

Next, we consider explicit and modified Euler's methods.

Algorithm 4: Explicit Euler's method for exercise 2

Data: number N of parts in which we divide the interval $[0, 1]$
Result: Sequence of values $u_0, u_1, \dots, u_n$
Function *Euler*(N)**:**
    $h := \frac{1}{N}$;
    for $k = 0 \,..\, N-1$ **do**
        $t_k = hk$;
        $u_{k+1} := u_k + h(1 + u_k)$;
    return $[u_0, u_1, \dots, u_N]$;

Algorithm 5: Modified Euler's method for exercise 2

Data: number N of parts in which we divide the interval $[0, 1]$
Result: Sequence of values $u_0, u_1, \dots, u_n$
Function *Euler*(N)**:**
    $h := \frac{1}{N}$;
    for $k = 0 \,..\, N-1$ **do**
        $t_k = hk$;
        $v_k := u_k + h/2(1 + u_k)$;
        $u_{k+1} := u_k + h(1 + v_k)$;
    return $[u_0, u_1, \dots, u_N]$;

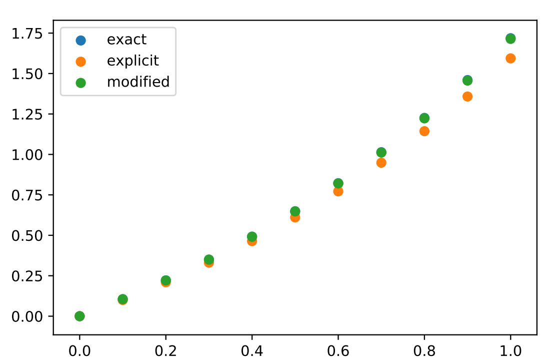
Applying these algorithms with $N = 10$ and $h = 0.1$, we obtain the approximate values.

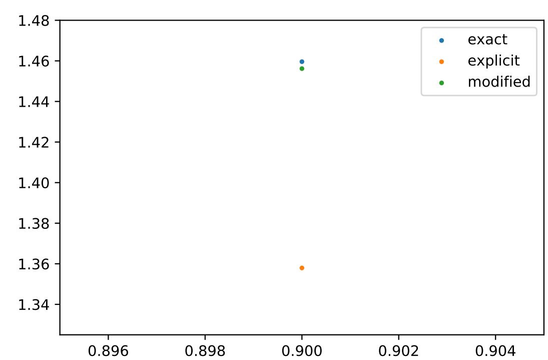
Attention: I made a computational error during the course and when I substituted v_k into the expression for u_{k+1}, after expanding the brackets, I obtained a wrong expression. Please be careful with computations.

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Exact	0.0	0.105	0.221	0.349	0.491	0.648	0.822	1.013	1.225	1.459	1.718
Modified	0.0	0.105	0.221	0.349	0.490	0.647	0.820	1.011	1.222	1.456	1.714
Explicit	0.0	0.1	0.210	0.331	0.464	0.610	0.771	0.948	1.143	1.357	1.593

Table 2: Comparing different methods

After finishing this table we see that version *modified* has much better performance compared to explicit iteration. We reinterpret this result on the plot.





In addition, we can compare the errors made by explicit and modified method.

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Modified	0.0	-0.0001	-0.0003	-0.0006	-0.0009	-0.0012	-0.0016	-0.0021	-0.0027	-0.0034	-0.0042
Explicit	0.0	-0.005	-0.011	-0.018	-0.027	-0.038	-0.050	-0.065	-0.081	-0.101	-0.124

Table 3: Comparing the errors

□

Exercise 6. On considère le problème suivant :

$$\begin{cases} y'(t) - \cos(t)y(t) = 0, & t \in [0,1] \\ y(0) = 1 & . \end{cases}$$

(*i*) Écrire ce problème sous la forme d'un problème de Cauchy. Donner, en particulier, la fonction $f(t, y)$, et les valeurs de t_0 et de y_0;

(*ii*) Montrer que ce problème admet une solution unique. Donner l'expression explicite de cette solution, si possible.

(*iii*) Donner l'équation d'itération de la méthode d'Euler explicite, en général et dans le cas particulier de ce problème de Cauchy.

(*iv*) Donner l'équation d'itération de la méthode d'Euler implicite, en général et dans le cas particulier de ce problème de Cauchy. Si cest possible, donnez l'expression de u_{i+1} en fonction de h, u_i et t_{i+1}.

(*v*) Appliquer la méthode d'Euler implicite avec $n = 2$ au problème ci-dessus.

(*vi*) Écrivez une fonction Matlab (sur papier) dont les arguments sont un nombre `u0` et un nombre entier `N`. Cette fonction devra calculer et afficher les nombres $u_1, u_2, \dots, u_N$ qui approchent la fonction y par la méthode d'Euler implicite.

Solution. I will skip certain aspects to make the solution as short as possible.

First, we try to find (heuristially) the explicit solution.

$$\frac{dy}{dt} = y \cos t$$

After variable splitting

$$\frac{dy}{y} = \cos t dt$$

Integration yields

$$\ln y = \sin t + C$$

So that (combined with the initial conditions $y(0) = 1$)

$$y = e^{\sin t}.$$

In order to prove the uniqueness of the solution, we use again theorem of Cauchy–Lipschitz. In the Cauchy form (1), the function $f(t, y)$ takes form

$$f(t, y) = y \cdot \cos(t)$$

This function is Lipschitz-continuous with a constant 1 with respect to its second variable:

$$|f(t, y_1) - f(t, y_2)| = |\cos t| \cdot |y_1 - y_2| \le 1 \cdot |y_1 - y_2|.$$

Therefore, the solution is unique.

Let us focus on the implicit method first. The equation for the implicit method is

$$u_{k+1} = u_k + hu_{k+1} \cos t_{k+1},$$

which is a linear equation with respect u_{k+1} and therefore, can be explicitly solved:

$$u_{k+1} = \frac{u_k}{1 - h \cos t_{k+1}},$$

if $h \cos t_{k+1} \neq 1$.

Therefore, we can provide algorithms for both explicit and implicit Euler's method for this particular problem.

Algorithm 6: Explicit Euler's method for exercise 6

Data: number N of parts in which we divide the interval $[0, 1]$
Result: Sequence of values $u_0, u_1, \ldots, u_n$
Function *Euler*(N)**:**

    $h := \frac{1}{N}$;
    for $k = 0 \mathrel{..} N - 1$ **do**
        $t_k = hk$;
        $u_{k+1} := u_k(1 + \cos t_k)$;
    return $[u_0, u_1, \ldots, u_N]$;

Algorithm 7: Implicit Euler's method for exercise 6

Data: number N of parts in which we divide the interval $[0, 1]$
Result: Sequence of values $u_0, u_1, \ldots, u_n$
Function *Euler*(N)**:**

    $h := \frac{1}{N}$;
    for $k = 0 \mathrel{..} N - 1$ **do**
        $t_k = hk$;
        $u_{k+1} := \frac{u_k}{1 - h \cos t_{k+1}}$;
    return $[u_0, u_1, \ldots, u_N]$;

I leave the numerical part and the Matlab code as an exercise (I have already presented the pseudocode for all of the algorithms, so the Matlab implementation is straightforward modulo syntax).

Remark 2. With some experience, one can notice that the expressions $1 + \cos t_k$ and $\frac{1}{1 - \cos t_{k+1}}$ correspond to a very similar entity, when t_k is small. This happens because

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots \approx 1 + x$$

when x is close to zero.

□

4 Questions and answers

Question 1. Can we say something about the convergence speed of the Euler methods? Is it linear or quadratic?

Answer. In this course, the notion of the speed of convergence for Euler methods is not discussed. However, the theoretical guarantee (which more or less meets practice) roughly corresponds to something much worse than linear speed. The error at each step is or order $O(h)$ for implicit and explicit methods, and of smaller order (say $O(h^2)$) for more precise methods, and the best we can obtain is something like $O(h^4)$ or $O(h^5)$ for a family of Runge–Kutta methods. This is beyond the scope of the first semester. Even for the fastest method, the error accumulates over the segments, so if you want to get higher precision, you need to have as many intervals as possible. □

Question 2. How to choose the branch for implicit method if there are several solutions?

Answer. If you have several solutions, by default you are not able to choose the correct branch without additional assumptions (and more advanced theoretical knowledge). At this stage it is proposed to you that you (heuristically) choose the branch y_{k+1} that is closer to the previous value y_k. □

Question 3. Is modified Euler method with step h the same as explicit Euler with half-step $h/2$?

Answer. No. □

Question 4. Which level of detalisation is expected when it is asked to write Matlab code on paper?

Answer. You are not machines, so if there are some small occasional errors, this is forgivable. □

Question 5. What to do if Cauchy–Lipschitz theorem doesn't help to prove the uniqueness of the solution?

Answer. In this case you cannot do anything to prove the uniqueness. Some other theoretical tools exist, but they are beyond the scope of the current course. □